

ON THE HIGHER FITTING IDEALS OF IWASAWA MODULES OF IDEAL CLASS GROUPS OVER REAL ABELIAN FIELDS

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ABSTRACT. Kurihara established a refinement of the minus-part of the Iwasawa main conjecture for totally real number fields using the higher Fitting ideals in his paper [Ku]. In this paper, by using Kurihara's methods and Mazur-Rubin theory, we study the higher Fitting ideals of the plus-part of Iwasawa modules associated to the cyclotomic \mathbb{Z}_p -extension of abelian fields for an odd prime number p . We define the higher cyclotomic ideals $\{\mathfrak{C}_i\}_{i \geq 0}$, which are ideals of the Iwasawa algebra defined by the Kolyvagin derivative classes of circular units, and prove that they give upper and lower bounds of the higher Fitting ideals in some sense, and determine the pseudo-isomorphism classes of the plus-part of Iwasawa modules. Our result can be regarded as an partial analogue of Kurihara's results and a refinement of the plus-part of the Iwasawa main conjecture for abelian fields.

1. INTRODUCTION

The Iwasawa main conjecture describes the characteristic ideals of certain Iwasawa modules. The characteristic ideals are important invariants on the structure of finitely generated torsion Iwasawa modules, but they are not enough to determine the pseudo-isomorphism classes of Iwasawa modules (cf. §2) completely.

The higher Fitting ideals have more detailed information on Iwasawa modules. For instance, the higher Fitting ideals determine the pseudo-isomorphism class and the least cardinality of generators of finitely generated torsion Iwasawa modules. (See Remark 2.3 and Remark 2.4.) In [Ku], Kurihara proved that all the higher Fitting ideals of the minus-part of the Iwasawa modules associated to the cyclotomic \mathbb{Z}_p -extension of certain CM-fields coincide with the higher Stickelberger ideals, which are defined by analytic objects arising from p -adic L -functions (cf. [Ku] Theorem 1.1). His result is a refinement of the minus-part of the Iwasawa main conjecture for totally real number fields.

Here, we study the higher Fitting ideals of the plus-part of the Iwasawa modules by similar methods as in [Ku]. In this paper, we construct a collection $\{\mathfrak{C}_{i,\chi}\}_{i \geq 0}$ of ideals of the Iwasawa algebra Λ_χ , which is an analogue of Kurihara's higher Stickelberger ideals, and prove that the ideals $\mathfrak{C}_{i,\chi}$ give upper and lower bounds of the higher Fitting ideals of the plus-part in some sense. (In certain cases, the ideals $\mathfrak{C}_{i,\chi}$ determine the pseudo-isomorphism class of the plus-part.) The main tool in [Ku] is the Kolyvagin system of Gauss sums. Instead, in this paper, we use the Euler system of circular

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units, so we can only treat the Iwasawa modules associated to the cyclotomic \mathbb{Z}_p -extension of subfields of cyclotomic fields.

In order to state the main theorem of this paper, we set notations in this paper. We fix an odd prime number p . Let $\overline{\mathbb{Q}}$ be an algebraic closure of \mathbb{Q} and K a totally real subfield of $\overline{\mathbb{Q}}$, which is a finite abelian extension of \mathbb{Q} . We assume that the prime number p is unramified in K/\mathbb{Q} . Let μ_n be the group of all n -th roots of unity contained in $\overline{\mathbb{Q}}$. For an integer m with $m \geq 0$, let F_m be the maximal totally real subfield of $K(\mu_{p^{m+1}})$ and $F_\infty := \bigcup_{m \geq 0} F_m$. We put $\Gamma_{m,n} := \text{Gal}(F_m/F_n)$ and $\Gamma_m := \text{Gal}(F_\infty/F_m)$. Especially, we write $\Gamma := \Gamma_0$. We fix a topological generator $\gamma \in \Gamma_0$. Let $\Lambda := \mathbb{Z}_p[[\text{Gal}(F_\infty/\mathbb{Q})]] = \varprojlim \mathbb{Z}_p[\text{Gal}(F_m/\mathbb{Q})]$.

Put $\Delta := \text{Gal}(F_0/\mathbb{Q}) = \Delta_0 \times \Delta_p$, where Δ_0 is the maximal subgroup of Δ whose order is prime to p , and Δ_p is the p -Sylow subgroup of Δ . We denote D_p the decomposition subgroup of Δ at p . (Note that D_p is uniquely determined since Δ is abelian.) We put $\widehat{\Delta} := \text{Hom}(\Delta, \overline{\mathbb{Q}}_p^\times)$. For any character $\chi \in \widehat{\Delta}$, we denote by \mathcal{O}_χ the $\mathbb{Z}_p[\Delta]$ -algebra, which is isomorphic to $\mathbb{Z}_p[\text{Im } \chi]$ as a \mathbb{Z}_p -algebra, and Δ acts on via χ . We denote the Λ -algebra $\mathcal{O}_\chi[[\Gamma_0]]$ by Λ_χ , and we identify Λ_χ with $\mathcal{O}_\chi[[T]]$ by the isomorphism $\Lambda_\chi \simeq \mathcal{O}_\chi[[T]]$ of \mathcal{O}_χ algebras defined by $\gamma \mapsto 1 + T$.

For any Λ -module M , we put $M_\chi := M \otimes_\Lambda \Lambda_\chi$. We define a Λ -module $X := \varprojlim A_m$, where $A_m := A_{F_m}$ is the p -Sylow subgroup of the ideal class group of F_m and the projective limit is taken with respect to the norm maps. It is well-known that the Λ_χ -module X_χ is finitely generated and torsion. In this paper, we study the higher Fitting ideals $\text{Fitt}_{\Lambda_\chi, i}(X_\chi)$ of X_χ for any non-trivial character $\chi \in \widehat{\Delta}$. Let $X_{\chi, \text{fin}}$ be the largest pseudo-null Λ_χ -submodule of X , and $X'_\chi := X_\chi / X_{\chi, \text{fin}}$. We treat X'_χ instead of X_χ in order to apply Kurihara's Euler system arguments, which work for finitely generated torsion Λ -modules whose structures are given by square matrices (cf. Lemma 2.7 and §7).

Comparing our setting with the minus part, which Kurihara studied, in the case of the plus-part, a problem lies in how to define the ideals which are substitutes for the higher Stickelberger ideals by Kurihara because we do not have elements as the Stickelberger elements in group rings of Galois groups. A key idea of this paper lies in the definition of ideal $\mathfrak{C}_{i, \chi}$ of Λ , called *the higher cyclotomic ideals* for each $i \in \mathbb{Z}_{\geq 0}$ which match Kurihara's arguments well. We shall define these ideals in §4, by using the Euler system of circular units (cf. Definition 4.15). Roughly speaking, first, we shall define the ideals $\mathfrak{C}_{i, m, N, \chi}$ of the group ring $R_{m, N, \chi} := \mathbb{Z}/p^N[\text{Gal}(F_m/\mathbb{Q})]_\chi$ generated by images of certain Kolyvagin derivatives $\kappa_{m, N}(\xi)$ by *all* $R_{m, N, \chi}$ -homomorphisms $(F_m^\times / (F_m^\times)^{p^N})_\chi \longrightarrow R_{m, N, \chi}$, then we shall define $\mathfrak{C}_{i, \chi}$ by the projective limit of them.

Let I and J be ideals of Λ_χ . Then, we write $I \prec J$ if there exists a height two ideal A of Λ_χ (called an "error factor") satisfying $AI \subseteq J$. Note that for two ideals I and J of Λ_χ , we have $I \prec J$ if and only if $I\Lambda_{\chi, \mathfrak{p}} \subseteq J\Lambda_{\chi, \mathfrak{p}}$ for all prime ideals \mathfrak{p} of height 1, where we denote the localization of Λ_χ at \mathfrak{p} by $\Lambda_{\chi, \mathfrak{p}}$. We write $I \sim J$ if $I \prec J$ and $J \prec I$. The relation \sim is an equivalence relation on ideals of Λ_χ .

The following theorem is a rough form of our main theorem in this paper.

Theorem 1.1. *We assume that the extension degree of K/\mathbb{Q} is prime to p . Let $\chi \in \widehat{\Delta}$ be a character satisfying $\chi(p) \neq 1$. Then, we have*

$$\text{Fitt}_{\Lambda_\chi, i}(X_\chi) \sim \mathfrak{C}_{i, \chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$. Moreover, we have

$$(1) \quad \text{ann}_{\Lambda_\chi}(X_{\chi, \text{fin}}) \text{Fitt}_{\Lambda_\chi, i}(X'_\chi) \subseteq \mathfrak{C}_{i, \chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$.

Remark 1.2. Let K/\mathbb{Q} and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. By a property of the principal (the 0-th) Fitting ideals, we have

$$\text{Fitt}_{\Lambda_\chi, 0}(X_{\chi, \text{fin}}) \subseteq \text{ann}_{\Lambda_\chi}(X_{\chi, \text{fin}}).$$

Note that we have

$$\text{Fitt}_{\Lambda_\chi, 0}(X_{\chi, \text{fin}}) \text{Fitt}_{\Lambda_\chi, 0}(X'_\chi) = \text{Fitt}_{\Lambda_\chi, 0}(X_\chi)$$

since X'_χ is a Λ_χ -module of projective dimension one (cf. 2.8). So, we have

$$\text{Fitt}_{\Lambda_\chi, 0}(X_\chi) \subseteq \mathfrak{C}_{0, \chi}.$$

Remark 1.3. In this paper, we also study the upper bounds of the higher Fitting ideals of the plus-part when $\Delta_p \neq 0$ or $\chi(p) = 1$. For the precise statement of our main theorem for upper bounds on the Fitting ideals, including these cases, see Theorem 7.1.

Theorem 1.1 implies that the higher cyclotomic ideals give “true” upper bounds in some special cases.

Corollary 1.4. *Let K/\mathbb{Q} and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. Assume that X_χ has no non-trivial Λ_χ -submodules. Then, we have*

$$\text{Fitt}_{\Lambda_\chi, i}(X_\chi) \subseteq \mathfrak{C}_{i, \chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$.

Remark 1.5. All known examples of X_χ is pseudo-null (cf. Greenberg conjecture, for example, see [Gree1] Conjecture 3.4), so we have no non-trivial example for Corollary 1.4 in present.

Remark 1.6. We prove the inequality

$$\text{ann}_{\Lambda_\chi}(X_{\chi, \text{fin}}) \text{Fitt}_{\Lambda_\chi, i}(X'_\chi) \subseteq \mathfrak{C}_{i, \chi}$$

in §7 by the Euler system argument using analogues of Kurihara’s elements. (See Theorem 7.1 and Corollary 7.2.) Then, in §8, we prove the inequalities

$$\text{Fitt}_{\Lambda_\chi, i}(X'_\chi) \succ \mathfrak{C}_{i, \chi} \quad (i \in \mathbb{Z}_{\geq 0})$$

by using the results of Mazur-Rubin theory on Kolyvagin systems. (See Theorem 8.2.) Note that the following is already known.

- Without Kurihara's elements, we can obtain (non-explicit) estimates

$$\text{Fitt}_{\Lambda_\chi, i}(X'_\chi) \prec \mathfrak{C}_{i, \chi} \quad (i \in \mathbb{Z}_{\geq 0}),$$

which are weaker than the estimates (1) in Theorem 1.1, by using usual Euler system argument and the Iwasawa main conjecture. (See Remark 8.13)

- By the Mazur-Rubin theory in [MR] §5, it turns out that the principal Kolyvagin systems of $(\Lambda_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1), \mathcal{F}_\Lambda)$ completely know the pseudo-isomorphism class of X_χ . (See Theorem 8.11, Corollary 8.20 and Corollary 8.23.) But the results in [MR] does not give explicit estimates of higher Fitting ideals of X_χ in terms of ideal.

The first assertion of Theorem 1.1 implies that the higher cyclotomic ideals translates the pseudo-isomorphism classes of X_χ into the terms of *ideals* of Λ_χ (cf. Remark 2.4). What is essentially new in this paper is the definition of the higher cyclotomic ideals and to give stronger estimates (1) of higher Fitting ideals which contains more refined information than the pseudo-isomorphism class of X_χ by using Euler system arguments via Kurihara arguments. The key slogan is that the usual Euler system arguments work well only when the relation matrix is diagonal, but the usual Euler system arguments via Kurihara's elements work well even when the relation matrix is square.

We remark on the relation between higher cyclotomic ideals and the structure of $A_{0, \chi} := A_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi$. By Mazur-Rubin theory in [MR], the isomorphism class of the \mathcal{O}_χ -module $A_{0, \chi}$ is determined by the Kolyvagin systems of $G_{\mathbb{Q}}$ -module $\mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$. By comparing Mazur-Rubin theory and higher cyclotomic ideals, we obtain the following proposition. (See Theorem 8.12 and Corollary 8.14.)

Proposition 1.7. *Let K/\mathbb{Q} and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. Then, the following holds.*

- (1) *The image of $\mathfrak{C}_{i, \chi}$ in the ring*

$$R_{0, \chi} := \mathbb{Z}_p[\text{Gal}(F_0/\mathbb{Q})]_\chi = \Lambda_\chi/(\gamma - 1) = \varprojlim_N R_{0, N, \chi} \simeq \mathcal{O}_\chi$$

coincides with the ideal $\mathfrak{C}_{i, F_0, \chi} := \varprojlim_N \mathfrak{C}_{i, 0, N, \chi}$ for any $i \in \mathbb{Z}_{\geq 0}$.

- (2) *We have $\text{Fitt}_{\mathcal{O}_\chi, i}(A_{0, \chi}) = \mathfrak{C}_{i, F_0, \chi}$ for any $i \in \mathbb{Z}_{\geq 0}$.*

So, by Nakayama's lemma, we obtain the following corollary. (See Corollary 8.15.)

Corollary 1.8. *Let K/\mathbb{Q} and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. Let r be a non-negative integer. Then, the following two properties are equivalent.*

- (1) *The least cardinality of generators of the Λ_χ -module X_χ is r .*
- (2) *$\mathfrak{C}_{r-1, \chi} \neq \Lambda_\chi$ and $\mathfrak{C}_{r, \chi} = \Lambda_\chi$.*

In §2 we recall the definition and some basic properties of higher Fitting ideals. In §3, we recall some preliminary results on Iwasawa theory. In §4, we define the higher Fitting ideals, and prove our main theorem for $i = 0$ (Theorem 4.16). In §5, we recall some basic facts on the Kolyvagin derivatives of the Euler system of circular

units, and induce some elements $x_{m,N}(n)_\chi$ of $(F_m^\times/p^N)_\chi$, are analogue of Kurihara's elements in [Ku]. The elements $x_{m,N}(n)_\chi$ play important roles in the Kurihara's Euler system arguments in the proof of Theorem 1.1. Especially, Proposition 5.6 is one of the keys of the Euler system arguments. In §6, we prove Proposition 6.1, which is a key proposition in the induction arguments in Euler system arguments. In §7, by the Euler system argument using analogues of Kurihara's elements, we prove the estimate

$$\text{ann}_{\Lambda_\chi}(X_{\chi,\text{fin}}) \text{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$$

for any $i \geq 0$ (see Theorem 7.1). We also treat the case $\Delta_p \neq 0$ or $\chi(p) = 1$. In §8, we compare the higher Fitting ideals with Mazur-Rubin theory. We apply theory on Kolyvagin systems established by Mazur and Rubin, and we prove Theorem 8.12, which is a result on the ground level, and the remaining part of Theorem 1.1.

Notation. In this paper, we use the following notation.

Let F be a perfect field, and \overline{F} an algebraic closure. We denote the absolute Galois group of F by $G_F := \text{Gal}(\overline{\mathbb{Q}}/F)$. For a topological abelian group T with continuous G_F action, let $H^*(F, T) = H^*(G_F, T)$ be the continuous Galois cohomology group.

In this paper, an algebraic number field is a subfield F of a fixed algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} such that the extension degree of F/\mathbb{Q} is a finite. For a finite set Σ of places of \mathbb{Q} , we denote by \mathbb{Q}_Σ the maximal extension field of \mathbb{Q} unramified outside Σ . For any algebraic number field F , we denote the ring of integers of F by \mathcal{O}_F , and the p -Sylow subgroup of the ideal class group of F by A_F .

We define $\mathbb{Q}_\infty/\mathbb{Q}$ to be the cyclotomic \mathbb{Z}_p -extension. For any $m \in \mathbb{Z}_{\geq 0}$, we denote by \mathbb{Q}_m the unique subfield of \mathbb{Q}_∞ whose extension degree over \mathbb{Q} is p^m . Note that the field F_m is the composite field of \mathbb{Q}_m and F_0 for any $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. We identify $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ as $\Gamma = \text{Gal}(F_\infty/F_0)$ by the natural isomorphism $\Gamma \xrightarrow{\cong} \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$.

Let L/K be a finite Galois extension of algebraic number fields. Let λ be a prime ideal of K , and λ' a prime ideal of L above λ . We denote the completion of K at λ by K_λ . If λ is unramified in L/K , the arithmetic Frobenius at λ' is denoted by $(\lambda', L/K) \in \text{Gal}(L/K)$. We fix a family of embeddings $\{\ell_{\overline{\mathbb{Q}}}: \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\}_{\ell:\text{prime}}$ satisfying the condition (Chb) as follows:

(Chb) *For any subfield $F \subset \overline{\mathbb{Q}}$ which is a finite Galois extension of \mathbb{Q} and any element $\sigma \in \text{Gal}(F/\mathbb{Q})$, there exist infinitely many prime numbers ℓ such that ℓ is unramified in F/\mathbb{Q} and $(\ell_F, F/\mathbb{Q}) = \sigma$, where ℓ_F is the prime ideal of \mathcal{O}_F corresponding to the embedding $\ell_{\overline{\mathbb{Q}}}|_F$.*

The existence of a family satisfying the condition (Chb) follows easily from the Chebotarev density theorem.

Let ℓ be a prime number. For an algebraic number field F , let ℓ_F be the prime ideal of F corresponding to the embedding $\ell_{\overline{\mathbb{Q}}}|_F$. Then, if $L \supseteq F$ is an extension of algebraic number fields, we have $\ell_L|\ell_F$.

For an abelian group M and a positive integer n , we write M/n in place of M/nM for simplicity. In particular, for the multiplicative group K^\times of a field K , we write K^\times/p^N in place of $K^\times/(K^\times)^{p^N}$. We denote by M_{tor} the kernel of the natural homomorphism $M \longrightarrow M \otimes \mathbb{Q}$.

For a Λ -module M , we denote the Γ_m -invariants (resp. Γ_m -coinvariants) of M by M^{Γ_m} (resp. M_{Γ_m}).

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2. HIGHER FITTING IDEALS

We use the same notation as in the previous section. In particular, we fix a finite abelian field K , and We define $\Lambda := \mathbb{Z}_p[[\text{Gal}(F_\infty/\mathbb{Q})]]$, where F_∞ is the maximal totally real subfield of $K(\mu_{p^\infty})$.

Here, we recall the definition and some basic properties of higher Fitting ideals briefly.

Definition 2.1 (higher Fitting ideals, see [No] §3.1). Let R be a commutative ring, and M a finitely presented R -module. Let

$$R^m \xrightarrow{f} R^n \longrightarrow M \longrightarrow 0$$

be an exact sequence of R -modules. For each $i \geq 0$, we define *the i -th Fitting ideal* $\text{Fitt}_{R,i}(M)$ as follows.

- When $0 \leq i < n$ and $m \geq n - i$, we define $\text{Fitt}_{R,i}(M)$ to be the ideal of R generated by all $(n - i) \times (n - i)$ minors of the matrix corresponding to f .
- When $0 \leq i < n$ and $m < n - i$, we define $\text{Fitt}_{R,i}(M) := 0$.
- When $i \geq n$, we define $\text{Fitt}_{R,i}(M) := R$.

The definition of these ideals depends only on M , and does not depend on the choice of the above exact sequence.

Remark 2.2. Let R be a commutative ring, S an R -algebra, and M a finitely presented R -module, Then, by definition of the higher Fitting ideals and the right exactness of tensor products, we have

$$\text{Fitt}_{S,i}(M \otimes_R S) = \text{Fitt}_{R,i}(M)S$$

for any $i \geq 0$.

Remark 2.3. Let R be a commutative ring, and M be a finitely presented R -module. If we have $\text{Fitt}_{R,i}(M) \neq R$, then the least cardinality of generators of M is greater than $i + 1$. Note that when R is a local ring or PID, the least cardinality of generators of M is $i + 1$ if and only if $\text{Fitt}_{R,i}(M) \neq R$ and $\text{Fitt}_{R,i+1}(M) = R$.

Remark 2.4. Fix an arbitrary character $\chi \in \widehat{\Delta}$, and let M and N be Λ_χ -modules. We say that M is pseudo-null if the order of M is finite. We write $M \sim_{\text{p.i.}} N$ if there exist homomorphism $M \rightarrow N$ whose kernel and cokernel are both pseudo-null, and we call M is pseudo-isomorphic to N . Note the relation $\sim_{\text{p.i.}}$ is an equivalence relation on finitely generated torsion Λ_χ -modules. Assume

$$M \sim_{\text{p.i.}} \bigoplus_{i=1}^n \Lambda_\chi / f_i \Lambda_\chi$$

and f_i divides f_{i+1} for $1 \leq i \leq n - 1$. Then, we have

$$\text{Fitt}_{\Lambda_\chi,i}(M) \sim \begin{cases} (\prod_{k=1}^{n-i} f_k) & (\text{if } i < n) \\ \Lambda_\chi & (\text{if } i \geq n) \end{cases}$$

for any non-negative integer i (cf. [Ku] Lemma 8.2). In particular, the pseudo-isomorphism class of M is determined by the higher Fitting ideals $\{\text{Fitt}_{\Lambda_\chi,i}(M)\}_{i \geq 0}$.

Remark 2.5. Let M be a finitely generated torsion Λ_χ -module. Then, the characteristic ideal $\text{char}_{\Lambda_\chi}(M)$ is the minimal principal ideal of λ_χ containing $\text{Fitt}_{\Lambda_\chi,0}(M)$.

Let us recall basic properties of higher Fitting ideals.

Lemma 2.6. *Let R be a commutative ring, and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence of finitely presented R -modules. Then, we have the following:

- (1) $\text{Fitt}_{R,i}(M) \subseteq \text{Fitt}_{R,i}(L)$ for any $i \geq 0$.
- (2) $\text{Fitt}_{R,i}(M) \subseteq \text{Fitt}_{R,i}(N)$ for any $i \geq 0$.
- (3) $\text{Fitt}_{R,i}(L) \text{Fitt}_{R,0}(N) \subseteq \text{Fitt}_{R,i}(M)$ for any $i \geq 0$.
- (4) $\text{Fitt}_{R,0}(L) \text{Fitt}_{R,i}(N) \subseteq \text{Fitt}_{R,i}(M)$ for any $i \geq 0$.

Proof. Consider free resolutions

$$R^s \xrightarrow{f} R^r \longrightarrow L \longrightarrow 0,$$

$$R^{s'} \xrightarrow{g} R^{r'} \longrightarrow N \longrightarrow 0$$

of R -modules L and N . Let $A \in M_{r,s}(R)$ (resp. $B \in M_{r',s'}(R)$) be the matrix associated to the R -linear map f (resp. g) for standard basis. Then, we have an exact sequence

$$R^{s+s'} \xrightarrow{h} R^{r+r'} \longrightarrow M \longrightarrow 0$$

such that the $(r + r') \times (s + s')$ matrix C associated to h is given by

$$C = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}.$$

All assertions of the lemma follows immediately from the computation of minors of the matrix C . \square

Later, we use the following lemma on principal Fitting ideals of Iwasawa modules.

Lemma 2.7 (for example, see [Ku] Theorem 8.1). *Let $R = \Lambda_\chi \simeq \mathcal{O}_\chi[[T]]$ and M a finitely generated torsion R -module. Suppose M contains no non-trivial pseudo-null R -submodule. Then, there exists an exact sequence*

$$0 \longrightarrow R^n \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

for some integer $n > 0$, and we have

$$\text{Fitt}_{R,0}(M) = \text{char}_R(M).$$

By lemma 2.7, we obtain the following corollary.

Corollary 2.8. *Let $R = \Lambda_\chi \simeq \mathcal{O}_\chi[[T]]$ and M a finitely generated torsion R -module. We denote the maximal pseudo-null R -submodule of M by M_{fin} . Then, we have*

$$\text{Fitt}_{R,0}(M) = \text{Fitt}_{R,0}(M_{\text{fin}}) \text{Fitt}_{R,0}(M/M_{\text{fin}}).$$

3. PRELIMINARIES

In this section, we recall some preliminary results on certain Iwasawa modules.

3.1. In this subsection, we give some remarks on “ χ -quotients” of Λ -modules. Recall we denote the p -component of $\Delta := \text{Gal}(F_0/\mathbb{Q})$ by Δ_p , and the maximal subgroup of Δ of order coprime to p by Δ_0 . Note that $\Lambda_{\chi_0} := \mathcal{O}_{\chi_0}[[\Gamma]][\Delta_p]$ is flat over Λ for any $\chi_0 \in \widehat{\Delta}_0$. In particular, if the extension degree of K/\mathbb{Q} is prime to p , then Λ_χ is flat over Λ for any $\chi \in \widehat{\Delta}$. When the degree of K/\mathbb{Q} is divisible by p , we have to treat such Λ -algebras more carefully.

Let $S_{\widehat{\Delta}}$ be a set of all representatives of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ -conjugacy classes of $\widehat{\Delta}$. We put

$$\iota_{S_{\widehat{\Delta}}} : \Lambda \longrightarrow \prod_{\chi} \Lambda_\chi$$

to be the natural homomorphism, where χ runs all elements of $S_{\widehat{\Delta}}$. Note that the cokernel of the homomorphism $\iota_{S_{\widehat{\Delta}}}$ is annihilated by $|\Delta_p|$. We use the following elementary lemma.

Lemma 3.1. *Let M be a Λ -module. Then, we consider a natural homomorphism*

$$\iota_{M, S_{\widehat{\Delta}}} : M \longrightarrow \prod_{\chi \in S_{\widehat{\Delta}}} M_\chi.$$

Then, the kernel and the cokernel of $\iota_{M, S_{\widehat{\Delta}}}$ are annihilated by $|\Delta_p|$.

Proof. We consider the exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{\iota_{S_{\widehat{\Delta}}}} \prod_{\chi} \Lambda_\chi \longrightarrow \prod_{\chi} \text{Coker } \iota_{S_{\widehat{\Delta}}} \longrightarrow 0.$$

Then, we obtain the exact sequence

$$\mathrm{Tor}_1^\Lambda(\mathrm{Coker} \iota_{S_{\widehat{\Delta}}}, M) \longrightarrow M \longrightarrow \prod_\chi M_\chi \longrightarrow \mathrm{Coker} \iota_{S_{\widehat{\Delta}}} \otimes_\Lambda M \longrightarrow 0.$$

Since the Λ -module $\mathrm{Coker} \iota_{S_{\widehat{\Delta}}}$ is annihilated by $|\Delta_p|$, the Λ -modules $\mathrm{Coker} \iota_{S_{\widehat{\Delta}}} \otimes_\Lambda M$ and $\mathrm{Tor}_1^\Lambda(\mathrm{Coker} \iota_{S_{\widehat{\Delta}}}, M)$ are annihilated by $|\Delta_p|$. \square

We denote the image of I_Δ in Λ_χ by $\bar{I}_{\Delta, \chi}$ for each character $\chi \in \widehat{\Delta}$.

Corollary 3.2. *Let M be Λ -modules with no non-zero \mathbb{Z}_p -torsion elements. We denote the Λ_χ -submodule of M_χ consisting of all \mathbb{Z}_p -torsion elements by $M_{\chi, \mathrm{tor}}$. Then, the Λ_χ -module $M_{\chi, \mathrm{tor}}$ is annihilated by $|\Delta_p|$.*

Proof. We consider the commutative diagram of natural homomorphisms

$$\begin{array}{ccc} M & \xrightarrow{f} & M \otimes \mathbb{Q} \\ \downarrow \iota_{M, S_{\widehat{\Delta}}} & & \downarrow \simeq \iota_{M, S_{\widehat{\Delta}}} \\ \prod_{\chi \in S_{\widehat{\Delta}}} M_\chi & \xrightarrow{\prod_\chi f_\chi} & \prod_{\chi \in S_{\widehat{\Delta}}} (M \otimes \mathbb{Q})_\chi. \end{array}$$

Then, the corollary follows from this commutative diagram and Lemma 3.1. \square

Corollary 3.3. *Let M and N be Λ -modules, and $f: M \longrightarrow N$ a homomorphism of Λ -modules. We consider the commutative diagram*

$$(2) \quad \begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow \iota_{M, S_{\widehat{\Delta}}} & & \downarrow \iota_{M, S_{\widehat{\Delta}}} \\ \prod_{\chi \in S_{\widehat{\Delta}}} M_\chi & \xrightarrow{\prod_\chi f_\chi} & \prod_{\chi \in S_{\widehat{\Delta}}} N_\chi \end{array}$$

induced by the homomorphism f .

- We have

$$\iota_{M, S_{\widehat{\Delta}}}(\mathrm{Ker} f) \supseteq |\Delta_p|^2 \cdot \mathrm{Ker} \left(\prod_\chi f_\chi \right).$$

In particular, for each character $\chi \in \widehat{\Delta}$, then $|\Delta_p|^2 \mathrm{Ker} f_\chi$ is contained in the image of the kernel of f in M_χ .

- The natural homomorphism $\mathrm{Coker}(f_\chi) \longrightarrow (\mathrm{Coker}(f))_\chi$ is an isomorphism of Λ_χ -modules for any character $\chi \in \widehat{\Delta}$.

Proof. The first assertion follows from the diagram (2) and Lemma 3.1. The second assertion is clear. \square

3.2. This and the next subsection, we recall some preliminary results on Iwasawa theory. In this section, we refer results on unit groups.

Let $m \in \mathbb{Z}_{\geq 0}$. We put $U_m := (\mathcal{O}_{F_m} \otimes \mathbb{Z}_p)^\times$ to be the group of semi-local units at p of F_m , and U_m^1 to be the maximal pro- p -part of U_m . We denote the group of units of \mathcal{O}_{F_m} by E_m , and the Sinnott's circular units in F_m by C_m (cf. [Si] §4). We define E_m^{cl} (resp. C_m^{cl}) to be the closure of E_m (resp. C_m) in U_m , and E_m^1 (resp. C_m^1) by $E_m^{\text{cl}} \cap U_m^1$ (resp. $C_m^{\text{cl}} \cap U_m^1$). We define $U_\infty := \varprojlim U_m^1$ and $E_\infty := \varprojlim E_m^1$, where these projective limit is taken with respect to the norm maps. Similarly, we define the limit $C_\infty := \varprojlim C_m^1$ of the projective system with respect to norm maps.

Remark 3.4. By Leopoldt's conjecture for abelian fields (cf. [Wa] Corollary 5.32), we have the natural isomorphism $E_m \otimes \mathbb{Z}_p \xrightarrow{\sim} E_m^1$. We also have the natural isomorphism $C_m \otimes \mathbb{Z}_p \xrightarrow{\sim} C_m^1$.

Proposition 3.5. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. There exists a homomorphism $\varphi: E_{\infty, \chi} \rightarrow \Lambda_\chi$ of Λ_χ -modules whose cokernel has finite order, and whose kernel is annihilated by p -power. (Note that if the extension degree of K/\mathbb{Q} is prime to p , then the E_∞ has no non-trivial p -torsion element.)*

Let $\chi \in \widehat{\Delta}$. We denote the restriction of χ to Δ_0 by χ_0 . We define an integer a_χ by

$$a_\chi = \begin{cases} 0 & \text{if } \chi_0(p) \neq 1; \\ 2 & \text{if } \chi_0(p) = 1. \end{cases}$$

For each $m \in \mathbb{Z}_{\geq 0}$, we consider the natural homomorphism $P_m^E: (E_\infty)_{\Gamma_m} \rightarrow E_m^1$. We define the homomorphism $P_m^F: (E_\infty)_{\Gamma_m} \rightarrow F_m^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to be the composition of P_m^E and the inclusion map $E_m^1 \simeq \mathcal{O}_{F_m}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \rightarrow F_m^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For each character $\chi \in \widehat{\Delta}$, the Λ_χ -homomorphisms P_m^E and P_m^F induce the homomorphisms

$$\begin{aligned} P_{m, \chi}^E: (E_{\infty, \chi})_{\Gamma_m} &\rightarrow E_m^1, \\ P_{m, \chi}^F: (E_{\infty, \chi})_{\Gamma_m} &\rightarrow (F_m^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi. \end{aligned}$$

Proposition 3.6. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. Then, there exist ideals $I_{P_\chi^E}$ and $J_{P_\chi^E}$ of Λ_χ of finite indices such that*

$$\begin{aligned} (\gamma - 1)^{a_\chi/2} |\Delta_p|^2 I_{P_\chi^E} \text{Ker } P_{m, \chi}^F &= (\gamma - 1)^{a_\chi/2} |\Delta_p|^2 I_{P_\chi^E} \text{Ker } P_{m, \chi}^F = \{0\}, \\ (\gamma - 1)^{a_\chi/2} J_{P_\chi^E} \text{Coker } P_{m, \chi}^E &= \{0\} \end{aligned}$$

for any $m \in \mathbb{Z}_{\geq 0}$.

Proof. For any cyclic group G and any G -module M , we denote the Tate cohomology groups by $\hat{H}^i(G, M)$. Fix a non-negative integer m . We have the exact sequence

$$0 \longrightarrow \varprojlim \hat{H}^{-1}(\Gamma_{m', m}, E_{m'}) \longrightarrow (E_\infty)_{\Gamma_m} \xrightarrow{P_m^E} E_m^1 \longrightarrow \varprojlim \hat{H}^0(\Gamma_{m', m}, E_{m'}) \longrightarrow 0$$

of Λ -modules. This exact sequence and Corollary 3.3 imply that

$$\begin{aligned} \text{Coker } P_{m,\chi}^E &= \varprojlim \hat{H}^0(\Gamma_{m',m}, E_{m'})_\chi, \\ \text{ann}_{\Lambda_\chi}(\text{Ker } P_{m,\chi}^E) &\supseteq |\Delta_p|^2 \text{ann}_{\Lambda_\chi}(\varprojlim \hat{H}^{-1}(\Gamma_{m',m}, E_{m'})_\chi). \end{aligned}$$

By Lemma 1.2 of [Ru1], there exists an integer k satisfying

$$\left| (\gamma - 1) \hat{H}^i(\Gamma_{m',m}, E_{m'}) \right| \leq p^k$$

for all $i \in \mathbb{Z}$ and all $m', m \in \mathbb{Z}_{\geq 0}$ with $m' \geq m$. (Note that the setting of Lemma 1.2 of [Ru1] seems to be different from ours, but the argument in the proof of this lemma works in more general situation containing our case.) Therefore, the assertion for characters $\chi \in \hat{\Delta}$ satisfying $\chi_0(p) = 1$ follows.

We assume that $\chi \in \hat{\Delta}$ is a character satisfying $\chi_0(p) \neq 1$. By Corollary 3.3, it is sufficient to show that for any $i \in \mathbb{Z}$, order of the Λ_{χ_0} -module

$$\hat{H}^i(\Gamma_{m',m}, E_{m'})_{\chi_0} = \hat{H}^i(\Gamma_{m',m}, (E_{m'} \otimes \mathbb{Z}_p)_{\chi_0})$$

is finite and bounded by a constant independent of m' .

Let m be an integer with $m' \geq m$. Since the $\mathbb{Z}_p[\text{Gal}(F_{m'}/\mathbb{Q})]$ -module $E_{m'}^1 \oplus \mathbb{Z}_p$ contains a submodule of finite index which is free of rank one, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_p[\text{Gal}(F_{m'}/\mathbb{Q})] \xrightarrow{f} E_{m'}^1 \oplus \mathbb{Z}_p \longrightarrow N \longrightarrow 0,$$

where N is a $\mathbb{Z}_p[\text{Gal}(F_{m'}/\mathbb{Q})]$ -module of finite order. This exact sequence induces the exact sequence

$$0 \longrightarrow \mathbb{Z}_p[\text{Gal}(F_{m'}/\mathbb{Q})]_{\chi_0} \xrightarrow{f_{\chi_0}} E_{m',\chi_0}^1 \oplus (\mathbb{Z}_p)_{\chi_0} \longrightarrow N_{\chi_0} \longrightarrow 0.$$

Note that we have $(\mathbb{Z}_p)_{\chi_0} = 0$ since χ_0 is non-trivial. We consider the Herbrand quotients, and obtain

$$\begin{aligned} \frac{\#\hat{H}^0(\Gamma_{m',m}, E_{m',\chi_0}^1)}{\#\hat{H}^{-1}(\Gamma_{m',m}, E_{m',\chi_0}^1)} &= \frac{\#\hat{H}^0(\Gamma_{m',m}, \mathbb{Z}_p[\text{Gal}(F_{m'}/\mathbb{Q})]_{\chi_0})}{\#\hat{H}^{-1}(\Gamma_{m',m}, \mathbb{Z}_p[\text{Gal}(F_{m'}/\mathbb{Q})]_{\chi_0})} \cdot \frac{\#\hat{H}^0(\Gamma_{m',m}, N_{\chi_0})}{\#\hat{H}^{-1}(\Gamma_{m',m}, N_{\chi_0})} \\ &= 1 \end{aligned}$$

Let $E_{m'}^{(p)}$ be the group of p -units of $F_{m'}$. Then, we have an exact sequence

$$(3) \quad 0 \longrightarrow E_{m'} \otimes \mathbb{Z}_p \xrightarrow{i} E_{m'}^{(p)} \otimes \mathbb{Z}_p \longrightarrow S_{m'} \longrightarrow 0,$$

where $S_{m'}$ is a $\mathbb{Z}_p[\text{Gal}(F_{m'}/\mathbb{Q})]$ -submodule of $\mathcal{I}_{F_{m'}}^p \otimes \mathbb{Z}_p$, which is a free \mathbb{Z}_p -module generated by all places of $F_{m'}$ above p . Note that the group D_p acts trivially on $S_{m'}$. So, the natural homomorphism

$$i_{\chi_0}: (E_{m'} \otimes \mathbb{Z}_p)_{\chi_0} \longrightarrow (E_{m'}^{(p)} \otimes \mathbb{Z}_p)_{\chi_0}$$

is an isomorphism. Then, we have

$$(4) \quad \frac{\#\hat{H}^0(\Gamma_{m',m}, (E_{m'}^{(p)} \otimes \mathbb{Z}_p)_{\chi_0})}{\#\hat{H}^{-1}(\Gamma_{m',m}, (E_{m'}^{(p)} \otimes \mathbb{Z}_p)_{\chi_0})} = \frac{\#\hat{H}^0(\Gamma_{m',m}, E_{m',\chi_0}^1)}{\#\hat{H}^{-1}(\Gamma_{m',m}, E_{m',\chi_0}^1)} = 1.$$

By Corollary in §5.4 of [Iw], there exists an integer r such that

$$\left| \hat{H}^{-1}(\Gamma_{m',m}, E_{m'}^1) \right| = \left| \hat{H}^{-1}(\Gamma_{m',m}, E_{m'}^{(p)}) \right| \leq p^r$$

for all $m', m \in \mathbb{Z}$ satisfying $m' \geq m \geq 0$. By the equality (4), we also have

$$\left| \hat{H}^0(\Gamma_{m',m}, E_{m',\chi_0}^1) \right| = \left| \hat{H}^0(\Gamma_{m',m}, E_{m',\chi_0}^{(p)}) \right| \leq p^r$$

for all $m', m \in \mathbb{Z}$ satisfying $m' \geq m \geq 0$. Therefore, order of the Λ_{χ_0} -modules $\hat{H}^{-1}(\Gamma_{m',m}, E_{m'}^1)_{\chi_0}$ and $\hat{H}^0(\Gamma_{m',m}, (E_{m'} \otimes \mathbb{Z}_p)_{\chi_0})$ are bounded by a constant independent of m' . This completes the proof of proposition. \square

We can prove a more refined proposition than Proposition 3.9 in the case of $\Delta_p = 0$ and $\chi(p) \neq 1$, by the similar argument to the proof of [Ru4] Theorem 7.6. (We have to replace X_∞ in [Ru4] to $\text{Gal}(M_\infty/F_\infty)$ and U_∞ in [Ru4] to our U_∞ , where M_∞ is the maximal pro- p extension field of F_∞ unramified outside the places above p .)

Proposition 3.7. *Assume that the extension degree of K/\mathbb{Q} is prime to p , and the character $\chi \in \hat{\Delta}$ satisfies $\chi(p) \neq 1$. Then, we can take $I_{P_\chi^E} = \Lambda_\chi$ and $J_{P_\chi^E} = \text{ann}_{\Lambda_\chi}(X_{\chi,\text{fin}})$.*

Remark 3.8. Note that the assertions of [Ru4] Theorem 7.6 are for imaginary quadratic base fields, but we can also prove Proposition 3.7 by direct application of [Ru4] Theorem 7.6 by the manner as follows. Let L be an imaginary quadratic field and denote the discriminant of L/\mathbb{Q} by $D_{L/\mathbb{Q}}$. Assume that $(pD_{K/\mathbb{Q}}, D_{L/\mathbb{Q}}) = 1$. Then F_0 and L are linearly disjoint over \mathbb{Q} . So, we have the natural isomorphism $\Delta \simeq \text{Gal}(F_0 L/L)$, and regard χ as a character of $\text{Gal}(F_0 L/L)$. We define $\tilde{X}_\infty := \varprojlim_m A_{F_m L}$ and $\tilde{E}_\infty := \varprojlim_m \mathcal{O}_{F_m L}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let

$$P_{m,\chi}^{\tilde{E}}: (\tilde{E}_\infty)_{\Gamma_m} \longrightarrow \mathcal{O}_{F_m L}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

be the natural map. We can set $(K_\infty/K, F, \chi)$ in [Ru4] to $(LF_\infty/L, F_m L, \chi)$ in our notation, and apply [Ru4] Theorem 7.6 (ii). Then, it follows that $\text{pr}_{\tilde{E},m}$ is injective, and

$$\text{Coker}(P_{m,\chi}^{\tilde{E}}) \simeq \tilde{X}_{\infty,\chi}^{\Gamma_m} \subseteq \tilde{X}_{\text{fin},\chi},$$

where $\tilde{X}_{\text{fin},\chi}$ is the maximal pseudo-null Λ_χ -submodule of $\tilde{X}_{\infty,\chi}$. Since p is odd, we obtain

$$\begin{aligned} \text{Ker}(P_{m,\chi}^E) &= \text{Ker}(P_{m,\chi}^{\tilde{E}})^{j=1} = 0, \\ \text{Coker}(P_{m,\chi}^E) &= \text{Coker}(P_{m,\chi}^{\tilde{E}})^{j=1} \simeq (\tilde{X}_{\infty,\chi}^{\Gamma_m})^{j=1} = X_{\infty,\chi}^{\Gamma_m} \subseteq X_{\text{fin},\chi}. \end{aligned}$$

This implies Proposition 3.7.

3.3. In this subsection, we recall some results on the ideal class groups and the statement of the Iwasawa main conjecture.

We denote the p -Sylow subgroup of the ideal class group of F_m by A_{F_m} . We define the Λ -module X by $X := \varprojlim A_{F_m}$, where the projective limits are taken with respect to the norm maps. Note that X is a finitely generated Λ -torsion module.

For the Iwasawa modules X , we need the following well-known results.

Proposition 3.9 (cf. [Wa] Lemma 13.15). *The following holds.*

- (i) *For each $m \in \mathbb{Z}_{\geq 0}$, the natural homomorphism $X_{\Gamma_m} \rightarrow A_{F_m}$ is surjective.*
- (ii) *There exist an Λ -submodule Y of X such that $(\gamma - 1)X \subseteq Y \subseteq X$, and the kernel of the canonical homomorphism $X_{\Gamma_m, \chi} \rightarrow A_{F_m, \chi}$ is annihilated by $I_A := \text{ann}_{\Lambda}(Y/(\gamma - 1)X)$ for any $m \in \mathbb{Z}_{\geq 0}$.*

In the case of $\Delta_p = 0$ and $\chi(p) \neq 1$, we have a more refined proposition than Proposition 3.9. The following Proposition 3.10 is proved by the similar way to [Ru4] Theorem 5.4 (i). (Note that as in Remark 3.8, we can also prove it by the direct application of [Ru4] Theorem 5.4 (i).)

Proposition 3.10. *Assume that the extension degree of K/\mathbb{Q} is prime to p , and the character $\chi \in \hat{\Delta}$ satisfies $\chi(p) \neq 1$. Then, the natural homomorphism*

$$X_{\Gamma_m, \chi} \longrightarrow A_{F_m, \chi}$$

is an isomorphism for any $m \in \mathbb{Z}_{\geq 0}$.

Here, we recall the statement of the plus-part of the Iwasawa main conjecture briefly:

The Λ -modules E_{∞} , C_{∞} and X are as above. Let $\chi \in \hat{\Delta}$ be an arbitrary character. Then, we have $\text{char}_{\Lambda_{\chi}}(X_{\chi}) = \text{char}_{\Lambda_{\chi}}((E_{\infty}/C_{\infty})_{\chi})$.

(See [CS], [MW], [Ru2], [Grei] Theorem 3.1 and loc. cit. Remark c), et al.) We use the Iwasawa main conjecture in the proof of our main results.

4. HIGHER CYCLOTOMIC IDEALS

In this section, we define ideals $\mathfrak{C}_{i, \chi}$ of Λ_{χ} for each $i \in \mathbb{Z}_{\geq 0}$ by using circular units, and prove Theorem 1.1 for $i = 0$.

4.1. Here, we define some special circular units in order to define ideals $\mathfrak{C}_{i, \chi}$.

We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and regard $\overline{\mathbb{Q}}$ as a subfield of \mathbb{C} . For each positive integer n , we define

$$\zeta_n := \exp(2\pi i/n) \in \overline{\mathbb{Q}} \subset \mathbb{C},$$

which is a primitive n -th root of unity. Note that we have $\zeta_{mn}^m = \zeta_n$ for any positive integers m and n .

For each integer $N > 0$, we define

$$\begin{aligned} \mathcal{S}_N &:= \{ \ell \mid \ell \text{ is a prime number splitting completely in } K(\mu_{p^N})/\mathbb{Q} \}, \\ \mathcal{N}_N &:= \left\{ \prod_{i=1}^r \ell_i \mid r \in \mathbb{Z}_{>0}, \ell_i \in \mathcal{S}_N \ (i = 1, \dots, r), \text{ and } \ell_i \neq \ell_j \text{ if } i \neq j \right\} \cup \{1\}. \end{aligned}$$

In particular, if $\ell \in \mathcal{S}_N$, then we have $\ell \equiv 1 \pmod{p^N}$.

Let m be a non-negative integer, and put $F := F_m$. We denote the conductor of F/\mathbb{Q} by $\mathfrak{f}_F = \mathfrak{f}_{F/\mathbb{Q}}$. For a positive integer n prime to \mathfrak{f}_F , we define $H_{F,n} := \text{Gal}(F(\mu_n)/F)$. For simplicity, we write $H_n := H_{\mathbb{Q},n}$. If n is decomposed in $n = \prod_{i=1}^r \ell_i^{e_i}$, where ℓ_1, \dots, ℓ_r are distinct prime numbers and $e_i > 0$ for each i , then we have natural isomorphisms

$$\begin{aligned} \text{Gal}(F(\mu_n)/\mathbb{Q}) &\simeq \text{Gal}(F/F_0) \times H_{F,n}, \\ H_{F,n} &\simeq H_n \simeq H_{\ell_1^{e_1}} \times \cdots \times H_{\ell_r^{e_r}}. \end{aligned}$$

We identify these groups by the canonical isomorphisms.

Definition 4.1. Let m be a non-negative integer, and n a positive integer prime to $p\mathfrak{f}_K$.

(i) For each $d \in \mathbb{Z}_{>1}$ dividing \mathfrak{f}_K , we define

$$\eta_m^d(n) := N_{\mathbb{Q}(\mu_{p^{m+1}nd})/\mathbb{Q}(\mu_{p^{m+1}nd}) \cap F_m(\mu_n)}(1 - \zeta_d^{p^{-m}} \zeta_{np^{m+1}}) \in F_m(\mu_n)^\times.$$

(ii) For each $a \in \mathbb{Z}$ with $(a, p) = 1$, we define

$$\eta_m^{1,a}(n) := N_{K(\mu_{p^{m+1}n})/F_m(\mu_n)}\left(\frac{1 - \zeta_n^{p^{-m}} \zeta_{p^{m+1}}^a}{1 - \zeta_n^{p^{-m}} \zeta_{p^{m+1}}}\right) \in F_m(\mu_n)^\times.$$

In this paper, we call the elements $\eta_m^d(n)$ and $\eta_m^{1,a}(n)$ the basic circular units of $F_m(\mu_n)$.

The following lemma is well-known, and easily verified.

Lemma 4.2. Let m be a non-negative integer, n a positive integer prime to $p\mathfrak{f}_K$, and ℓ a prime divisor of ℓ . Let $\eta_m(n)^\bullet$ be a basic circular units of $F_m(\mu_n)$. Then, the following holds.

(i) We have

$$N_{F_m(\mu_n)/F_m(\mu_n/\ell)}(\eta_m^\bullet(n)) = \eta_m^\bullet(n/\ell)^{1 - \text{Fr}_\ell^{-1}},$$

where Fr_ℓ is the arithmetic Frobenius element at ℓ in $\text{Gal}(F_m(\mu_n/\ell)/\mathbb{Q})$.

(ii) We have

$$N_{F_{m+1}(\mu_n)/F_m(\mu_n)}(\eta_{m+1}^\bullet(n)) = \eta_m^\bullet(n).$$

Remark 4.3. Let \mathcal{K} be a composite field of \mathbb{Q}_∞ and $\mathbb{Q}(\mu_n)$ for all positive integers n satisfying $(n, p\mathfrak{f}_{K/\mathbb{Q}}) = 1$. We fix a circular unit

$$\eta := \prod_{d|\mathfrak{f}_K} \eta_0^d(1)^{u_d} \times \prod_{i=1}^r \eta_0^{1,a_i}(1)^{v_i} \in F_0^\times,$$

where $r \in \mathbb{Z}_{>0}$, u_d and v_i are elements of $\mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})]$ for each positive integers d and i with $d|\mathfrak{f}_K$ and $1 \leq i \leq r$, and a_1, \dots, a_r are integers prime to p . For any non-negative integer m and any positive integer n satisfying $(n, \mathfrak{f}_{K/\mathbb{Q}}) = 1$, we put

$$\eta_m(n) := \prod_{d|\mathfrak{f}_K} \eta_1^d(n)^{u_d} \times \prod_{i=1}^r \eta_m^{1,a_i}(n)^{v_i} \in F_m^\times.$$

We also denote by $\eta_m(n)_\chi$ the image of $\eta_m(n)_\chi$ in $H^1(\mathbb{Q}_m, \mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1))$ by the natural homomorphism

$$(F_m(\mu_n)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi = H^1(\mathbb{Q}_m, \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1))_\chi \longrightarrow H^1(\mathbb{Q}_m, \mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)).$$

Then, the collection

$$\{\eta_m(n)_\chi \in H^1(\mathbb{Q}_m, \mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1))\}_{m,n}$$

of Galois cohomology classes defines an Euler system for $(\mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1), \mathcal{K}/\mathbb{Q}, p\mathfrak{f}_{K/\mathbb{Q}})$ in the sense of [Ru5].

In particular, Lemma 4.2 (ii) implies that $(\eta_m^d(1))_{m \geq 0}$ is a norm compatible system, so it is an element of C_∞ . Later, we use the following result.

Proposition 4.4 (See [Gre] Lemma 2.3). *The Λ -module C_∞ is generated by*

$$\{(\eta_m^d(1))_{m \geq 0} \mid d \in \mathbb{Z}_{>1}, d \nmid \mathfrak{f}_K\} \cup \{(\eta_m^{1,a}(1))_{m \geq 0} \mid a \in \mathbb{Z}, (a, p) = 1\}.$$

Moreover, in the case of $\Delta_p = 0$, the following result is known.

Proposition 4.5 ([Tsu] Lemma 6.2). *Assume that the extension degree of K/\mathbb{Q} is prime to p , and the character $\chi \in \widehat{\Delta}$ is non-trivial. Then, The Λ_χ -module $C_{\infty, \chi}$ is generated by one element. So, combining with Proposition 3.5, the Λ_χ -module $C_{\infty, \chi}$ is free of rank one.*

Remark 4.6. If the extension degree of K/\mathbb{Q} is prime to p , and a character $\chi \in \widehat{\Delta}$ satisfies $\chi(p) \neq 1$, then we can easily show that any circular unit $\eta_\chi \in C_{0, \chi}^1$ extends to an element

$$\{\eta_{m, \chi}\}_m \in C_{\infty, \chi} = \varprojlim_m C_{m, \chi}^1$$

satisfying $\eta_{0, \chi} = \eta_\chi$. This fact implies that any circular unit $\eta \in C_{0, \chi}^1$ extends to an Euler system $\{\eta_m(n)_\chi\}_{m, n}$ for $(\mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1), \mathcal{K}/\mathbb{Q}, p\mathfrak{f}_{K/\mathbb{Q}})$ which consists of Λ_χ -linear combination of basic circular units, and satisfies $\eta_0(1)_\chi = \eta_\chi$.

4.2. In this subsection, we define the higher cyclotomic ideals $\mathfrak{C}_{i, \chi}$ by using Kolyvagin derivatives $\kappa_{m, N}^\bullet(n)$ of Euler systems of circular units. First, let us recall the notion of Kolyvagin derivatives. Let ℓ be prime number contained in \mathcal{S}_N . We shall take a generator σ_ℓ of a cyclic group $H_\ell = \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ as follows. We put $N_{\{\ell\}} := \text{ord}_p(\ell - 1)$, where ord_p is the normalized additive valuation at p , namely, $\text{ord}_p(p) = 1$. Then, we have $N_{\{\ell\}} \geq N \geq 1$. By the fixed embedding $\ell_{\overline{\mathbb{Q}}}: \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, we regard $\mu_{p^{N_{\{\ell\}}}}$ as a subset of \mathbb{Q}_ℓ . We identify $\text{Gal}(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell)$ with $H_\ell = \text{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q})$ by the isomorphism defined by $\ell_{\overline{\mathbb{Q}}}$. Let F be the maximal p -extension field of \mathbb{Q} contained in $\mathbb{Q}(\mu_\ell)$, and π a uniformizer of F_{ℓ_F} . We fix a generator σ_ℓ of H_ℓ such that

$$\pi^{\sigma_\ell - 1} \equiv \zeta_{p^{N_{\{\ell\}}}} \pmod{\mathfrak{m}_\ell},$$

where \mathfrak{m}_ℓ is the maximal ideal of F_{ℓ_F} , and $\zeta_{p^{N_{\{\ell\}}}}$ is a primitive $p^{N_{\{\ell\}}}$ -th root of unity defined as above. Note that the definition of σ_ℓ does not depend on the choice of π .

Definition 4.7. (i) For $\ell \in \mathcal{S}_N$, we define

$$D_\ell := \sum_{k=1}^{\ell-2} k\sigma_\ell^k \in \mathbb{Z}[H_\ell].$$

(ii) Let $n = \prod_{i=1}^r \ell_i \in \mathcal{N}_N$, where $\ell_i \in \mathcal{S}_N$ for each i . Then, we define

$$D_n := \prod_{i=1}^r D_{\ell_i} \in \mathbb{Z}[H_n].$$

In order to define Kolyvagin derivatives of circular units, we use the following well-known lemma.

Lemma 4.8. *Let $n \in \mathcal{N}_N$. Then, for each $d \in \mathbb{Z}_{>1}$ dividing \mathfrak{f}_K and for each $a \in \mathbb{Z}$ prime to p , the images of $\eta_m^d(n)^{D_n}$ and $\eta_m^{1,a}(n)^{D_n}$ in $F_m(\mu_n)^\times/p^N$ are fixed by H_n .*

Note that $H^0(F_n(n), \mu_{p^N}) = 0$ in our situation, so by Kummer theory and Hochschild-Serre spectral sequence, the natural homomorphism

$$F_m^\times/p^N \longrightarrow (F_m(\mu_n)^\times/p^N)^{H_n}$$

is an isomorphism. By Lemma 4.8, we can define Kolyvagin derivatives $\kappa_{m,N}^\bullet(n)$ of (basic) circular units as follows.

Definition 4.9. Let $n \in \mathcal{N}_N$. For each $d \in \mathbb{Z}_{>1}$ dividing \mathfrak{f}_K (resp. $a \in \mathbb{Z}$ prime to p), we define

$$\kappa_{m,N}^d(n) \in F_m^\times/p^N \quad (\text{resp. } \kappa_{m,N}^{1,a}(n) \in F_m^\times/p^N)$$

to be the unique element whose image in $F_m(\mu_n)^\times/p^N$ is $\eta_m^d(n)^{D_n}$ (resp. $\eta_m^{1,a}(n)^{D_n}$).

Now, let us define the higher cyclotomic ideals $\{\mathfrak{C}_{i,\chi}\}_{i \geq 0}$. First, we fix integers m and N satisfying $N \geq m+1 > 0$. Let $\chi \in \widehat{\Delta}$, and put

$$\begin{aligned} R_{m,N} &:= \mathbb{Z}/p^N[\text{Gal}(F_m/\mathbb{Q})]; \\ R_{m,N,\chi} &:= R_{m,N} \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\chi \simeq \mathcal{O}_\chi/p^N[\Gamma_{m,0}]. \end{aligned}$$

Then, we have

$$R_{m,N} \simeq \Lambda_{\Gamma_m}/p^N, \quad R_{m,N,\chi} \simeq \Lambda_{\chi,\Gamma_m}/p^N,$$

where Λ_{Γ_m} (resp. Λ_{χ,Γ_m}) denotes the Γ_m -coinvariant of Λ (resp. Λ_χ). As in [Ku], we need the notion called *well-ordered*.

Definition 4.10. Let $n \in \mathcal{N}_N$. We call n *well-ordered* if and only if n has a factorization $n = \prod_{i=1}^r \ell_i$ with $\ell_i \in \mathcal{S}_N$ for each i such that ℓ_{i+1} splits in $F_m(\prod_{j=1}^i \ell_j)/\mathbb{Q}$ for $i = 1, \dots, r-1$. In other words, n is well-ordered if and only if n has a factorization $n = \prod_{i=1}^r \ell_i$ such that

$$\ell_{i+1} \equiv 1 \pmod{p^N \prod_{j=1}^i \ell_j}$$

for $i = 1, \dots, r-1$. We denote by $\mathcal{N}_N^{\text{w.o.}}$ the set of all elements in \mathcal{N}_N which are well-ordered.

Let $n \in \mathcal{N}_N^{\text{w.o.}}$ with the decomposition $n = \prod_{i=1}^r \ell_i$, where $\ell_i \in \mathcal{S}_N$ for each i . We put $\epsilon(n) := r$. We define $\mathcal{W}_{m,N,\chi}(n)$ to be the $R_{m,N,\chi}$ -submodule of $(F_m^\times/p^N)_\chi$ generated by the image of

$$\{\kappa_{m,a}^d(n) \mid d \in \mathbb{Z}_{>0} \text{ dividing } \mathfrak{f}_K\} \cup \{\kappa_m^{1,a}(n) \mid a \in \mathbb{Z} \text{ prime to } p\}.$$

We put $\mathcal{H}_{m,N,\chi} := \text{Hom}_{R_{m,N,\chi}}((F_m^\times/p^N)_\chi, R_{m,N,\chi})$.

Definition 4.11. We define $\mathfrak{C}_{i,m,N,\chi}$ to be the ideal of $R_{m,N,\chi}$ generated by

$$\bigcup_{f \in \mathcal{H}_{m,N,\chi}} \bigcup_n f(\mathcal{W}_{m,N,\chi}(n)),$$

where n runs through all elements of $\mathcal{N}_N^{\text{w.o.}}$ satisfying $\epsilon(n) \leq i$.

Remark 4.12. Note that the $R_{m,N,\chi}$ -module $\text{Hom}(\mathcal{O}_\chi/p^N[\Gamma_{m,0}], \mathbb{Q}_p/\mathbb{Z}_p)$ is injective and free of rank one. So, $R_{m,N,\chi}$ is an injective $R_{m,N,\chi}$ -module. In particular, the restriction map

$$\mathcal{H}_{m,N,\chi} \longrightarrow \mathcal{H}_{m,N,\chi}(n) := \text{Hom}_{R_{m,N,\chi}}(\mathcal{W}_{m,N,\chi}(n), R_{m,N,\chi})$$

is surjective. This implies that the ideal $\mathfrak{C}_{i,m,N,\chi}$ coincides with the ideal of $R_{m,N,\chi}$ generated by

$$\bigcup_n \bigcup_{f \in \mathcal{H}_{m,N,\chi}(n)} \text{Im}(f),$$

where $\text{Im}(f)$ is the image of f , and n runs through all elements of $\mathcal{N}_N^{\text{w.o.}}$ satisfying $\epsilon(n) \leq i$.

In order to define the higher cyclotomic ideals, we need the following lemma and its corollary. (Note that in this paper, we use the first assertion of the following lemma only when $n = 1$.)

Lemma 4.13. *Let m_1, m_2, N_1 and N_2 be integers satisfying $m_2 \geq m_1$ and $N_2 \geq N_1$. Take a positive integer n prime to $p\mathfrak{f}_{K/\mathbb{Q}}$. Then, the following holds.*

(i) *For any $R_{m_2,N_2,\chi}[H_n]$ -homomorphism*

$$f_2: (F_{m_2}(\mu_n)^\times/p^{N_2})_\chi \longrightarrow R_{m_2,N_2,\chi}[H_n],$$

there exists an $R_{m_2,N_2,\chi}[H_n]$ -homomorphism

$$f_1: (F_{m_1}(\mu_n)^\times/p^{N_1})_\chi \longrightarrow R_{m_1,N_1,\chi}[H_n]$$

which makes the diagram

$$\begin{array}{ccc} (F_{m_2}(\mu_n)^\times/p^N)_\chi & \xrightarrow{f_2} & R_{m_2,N_2,\chi}[H_n] \\ \downarrow N_{F_{m_2}/F_{m_1}} & & \downarrow \text{mod } (\gamma^{p^{m_1}} - 1, p^{N_1}) \\ (F_{m_1}(\mu_n)^\times/p^N)_\chi & \xrightarrow{f_1} & R_{m_1,N_1,\chi}[H_n] \end{array}$$

commute.

(ii) Assume $\Delta_p = 0$ and $N_1 = N_2 =: N$. Then, for any $R_{m_1, N_1, \chi}[H_n]$ -homomorphism

$$f_1: (F_{m_1}(\mu_n)^\times / p^N)_\chi \longrightarrow R_{m_1, N, \chi}[H_n],$$

there exists an $R_{m_2, N_2, \chi}[H_n]$ -homomorphism

$$f_2: (F_{m_2}(\mu_n)^\times / p^N)_\chi \longrightarrow R_{m_2, N_2, \chi}[H_n]$$

which makes the diagram

$$\begin{array}{ccc} (F_{m_2}(\mu_n)^\times / p^N)_\chi & \xrightarrow{f_2} & R_{m_2, N_2, \chi}[H_n] \\ \downarrow N_{F_{m_2}/F_{m_1}} & & \downarrow \text{mod } (\gamma^{p^{m_1}-1}, p^{N_1}) \\ (F_{m_1}(\mu_n)^\times / p^N)_\chi & \xrightarrow{f_1} & R_{m_1, N_1, \chi}[H_n] \end{array}$$

commute.

Proof. Let us prove the first assertion of the lemma. Note that we can easily reduce the proof of this claim to the following two cases:

- (1) $(m_2, N_2) = (m_1, N_1 + 1)$;
- (2) $(m_2, N_2) = (m_1 + 1, N_1)$.

In the case (1), our lemma is clear. We shall show the lemma in the case (2). We put $m = m_1$, $N = N_1 = N_2$, $R_1 = R_{m, N}[H_n]$, $R_2 = R_{m+1, N}[H_n]$, and the natural surjection $\text{pr}: R_2 \longrightarrow R_1$. We denote by ι_χ the natural homomorphism

$$\iota: (F_m(\mu_n)^\times / p^N)_\chi \longrightarrow (F_{m+1}(\mu_n)^\times / p^N)_\chi.$$

We define an element

$$N_{m+1/m} := \sum_{\sigma \in \text{Gal}(F_{m+1}/F_m)} \sigma \in R_2.$$

Then, there is a unique isomorphism

$$\nu_{m+1/m}: R_1 \xrightarrow{\simeq} N_{m+1/m} R_2 = (R_2)^{\Gamma_{m+1, m}}$$

of R_1 -modules satisfying $1 \mapsto N_{m+1/m}$. Let \mathcal{NF} be the image of $(F_{m+1}(\mu_n)^\times / p^N)_\chi$ in $(F_m(\mu_n)^\times / p^N)_\chi$ by the norm map. Note the composition map

$$\nu_{m+1/m} \circ \text{pr}: R_2 \longrightarrow R_2$$

coincides with the scalar multiplication by $N_{m+1/m}$, so there exist a unique R_2 -linear homomorphism

$$f_1^{(0)}: \iota_\chi(\mathcal{NF}) = N_{m+1/m} (F_{m+1}(\mu_n)^\times / p^N)_\chi \longrightarrow R_2$$

which makes the diagram

$$\begin{array}{ccccc}
 & (F_{m+1}(\mu_n)^\times/p^N)_\chi & \xrightarrow{f_2} & R_2 & \xrightarrow{\times N_{m/m+1}} R_2 \\
 & \downarrow \times N_{m+1,m} & & \downarrow \text{pr} & \nearrow \nu_{m+1/m} \\
 \mathcal{NF} & \xrightarrow{\iota_\chi} \iota_\chi(\mathcal{NF}) & \xrightarrow{f_1^{(0)}} & R_1 & \\
 & \nwarrow N_{F_m/F_{m+1}} & & &
 \end{array}$$

commute. By injectivity of R_1 , we can extend $f_1^{(0)} \circ \iota_\chi$ to be a homomorphism

$$f_1: (F_{m_1}(\mu_n)^\times/p^{N_1})_\chi \longrightarrow R_1$$

satisfying $f_1|_{\mathcal{NF}} = f_1^{(0)} \circ \iota$. This completes the proof of the first assertion.

Next, let us prove the second assertion. It is sufficient to show in the case of $(m_2, N_2) = (m_1 + 1, N_1)$. Here, we use the same notation as in the proof of the first assertion. Let $f_1: (F_{m_1}(\mu_n)^\times/p^N)_\chi \longrightarrow R_1$ be a given R_1 -homomorphism. Note that the natural homomorphism

$$\iota: F_m(\mu_n)^\times/p^N \longrightarrow F_{m+1}(\mu_n)^\times/p^N$$

is injective since we have $H^0(F_{m+1}(\mu_n), \mu_{p^N}) = 0$. We regard $(F_{m_1}(\mu_n)^\times/p^N)_\chi$ as an R_2 -submodule of $(F_{m+1}(\mu_n)^\times/p^N)_\chi$ by the natural injection ι_χ . Note R_2 is an injective R_2 -module, so we can extend the homomorphism

$$\nu_{m+1/m} \circ f_1: (F_m(\mu_n)^\times/p^N)_\chi \longrightarrow R_2$$

to an R_2 -homomorphism $f_2: (F_{m+1}(\mu_n)^\times/p^N)_\chi \longrightarrow R_2$. By definition of f_2 , we obtain the commutative diagram

$$\begin{array}{ccccc}
 (F_{m+1}(\mu_n)^\times/p^N)_\chi & \xrightarrow{f_2} & R_2 & \xrightarrow{\times N_{m+1/m}} & R_2 \\
 \downarrow \times N_{m+1/m} & & \downarrow \text{pr} & & \nearrow \nu_{m+1/m} \\
 (F_m(\mu_n)^\times/p^N)_\chi & \xrightarrow{f_1} & R_1 & &
 \end{array}$$

in the category of R_2 -modules. □

Let m_1, m_2, N_1 and N_2 be integers satisfying $N_1 \geq m_1 + 1, N_2 \geq m_2 + 1, m_2 \geq m_1$ and $N_2 \geq N_1$. Let $n \in \mathcal{N}_{N_2}^{\text{w.o.}}$ be an element satisfying $\epsilon(n) \leq i$. Note that it follows from Lemma 4.2 that the image of $\mathcal{W}_{m_2, N_2, \chi}(n)$ by the norm map

$$N_{F_{m_2}/F_{m_1}}: (F_{m_2}^\times/p^{N_2})_\chi \longrightarrow (F_{m_1}^\times/p^{N_1})_\chi$$

is contained in $\mathcal{W}_{m_1, N_1, \chi}(n)$. Hence the following corollary follows from Lemma 4.13.

Corollary 4.14. *Let m_1, m_2, N_1 and N_2 be integers satisfying $N_1 \geq m_1 + 1, N_2 \geq m_2 + 1, m_2 \geq m_1$ and $N_2 \geq N_1$. Then, the image of $\mathfrak{C}_{i, m_2, N_2, \chi}$ by the projection $R_{m_2, N_2, \chi} \longrightarrow R_{m_1, N_1, \chi}$ is contained in $\mathfrak{C}_{i, m_1, N_1, \chi}$. Moreover, if we assume $\Delta_p = 0$ and $N_1 = N_2 =: N$, then the image of $\mathfrak{C}_{i, m_2, N, \chi}$ in $R_{m_1, N, \chi}$ coincides with $\mathfrak{C}_{i, m_1, N, \chi}$.*

Now, we can define the higher cyclotomic ideals.

Definition 4.15. We define the i -th cyclotomic ideal $\mathfrak{C}_{i,\chi}$ to be the ideal of Λ_χ by

$$\mathfrak{C}_{i,\chi} := \varprojlim \mathfrak{C}_{i,m,N,\chi},$$

where the projective limit is taken with respect to the system of the natural homomorphisms $\mathfrak{C}_{i,m_2,N_2,\chi} \longrightarrow \mathfrak{C}_{i,m_1,N_1,\chi}$ for integers m_1, m_2, N_1 and N_2 satisfying $N_1 \geq m_1 + 1$, $N_2 \geq m_2 + 1$, $m_2 \geq m_1$ and $N_2 \geq N_1$.

4.3. We take a generator $\theta \in \Lambda_\chi$ of $\text{char}_{\Lambda_\chi}(E_{\infty,\chi}/C_{\infty,\chi})$. Then, we denote the ideal of Λ_χ generated by

$$\bigcup_{\varphi \in \text{Hom}_{\Lambda_\chi}(E_{\infty,\chi}, \Lambda_\chi)} \theta^{-1} \varphi(C_{\infty,\chi})$$

by I_C . Note that I_C is an ideal of Λ_χ of finite index. Moreover, by Proposition 4.5, we have $I_C = \Lambda_\chi$ if the extension degree of K/\mathbb{Q} is prime to p .

In the rest of this section, we will prove the following theorem, which is a part of Theorem 1.1 for $i = 0$.

Theorem 4.16. Let $\chi \in \widehat{\Delta}$ be any character. Then, we have

- (i) $\mathfrak{C}_{0,\chi} \subseteq \text{Fitt}_{\Lambda_\chi,0}(X'_\chi)$.
- (ii) $(\gamma - 1)^{a_\chi} |\Delta_p|^4 I_C I_{P_\chi^E} J_{P_\chi^E} \text{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$.

In order to prove Theorem 4.16, by Iwasawa main conjecture, it is enough to prove the following Proposition 4.17.

Proposition 4.17. Let χ be a non-trivial character in $\widehat{\Delta}$. Then,

- (1) $(\gamma - 1)^{a_\chi} |\Delta_p|^4 I_{P_\chi^E} J_{P_\chi^E} I_C \text{char}_{\Lambda_\chi}(E_{\infty,\chi}/C_{\infty,\chi}) \subseteq \mathfrak{C}_{0,\chi}$;
- (2) $\mathfrak{C}_{0,\chi} \subseteq \text{char}_{\Lambda_\chi}(E_{\infty,\chi}/C_{\infty,\chi})$.

Proof. Let us prove Proposition 4.17. First, we prove the first assertion. By Proposition 3.5, we can take a homomorphism $\varphi: E_{\infty,\chi} \longrightarrow \Lambda_\chi$ of finite index. This induces a homomorphism

$$\bar{\varphi}_{m,N,\chi}: (E_{\infty,\chi})_{\Gamma_m}/p^N \longrightarrow R_{m,N,\chi}.$$

We take arbitrary elements $\delta_1 \in I_{P_\chi^E}$ and $\delta_2 \in J_{P_\chi^E}$.

Lemma 4.18. Let $\mathcal{NO}_{m,N,\chi}$ be the image of the homomorphism

$$(E_{\infty,\chi})_{\Gamma_m}/p^N \longrightarrow (F_m^\times/p^N)_\chi$$

induced by the homomorphism $P_{m,\chi}^F$ defined in §3.2. Then, the kernel of the natural homomorphism is annihilated by $(\gamma-1)^{a_\chi} |\Delta_p|^4 I_{P_\chi^E} J_{P_\chi^E}$, and there exists a homomorphism $\psi: \mathcal{NO}_{m,N,\chi} \longrightarrow R_{m,N,\chi}$ which makes the diagram

$$\begin{array}{ccc} (C_{\infty,\chi})_{\Gamma_m}/p^N & \longrightarrow & (E_{\infty,\chi})_{\Gamma_m}/p^N \xrightarrow{(\gamma-1)^{a_\chi} |\Delta_p|^4 \delta_1 \delta_2 \cdot \bar{\varphi}_{m,N,\chi}} R_{m,N,\chi} \\ \downarrow & & \downarrow \\ \mathcal{W}_{m,N,\chi}(1) & \hookrightarrow & \mathcal{NO}_{m,N,\chi} \end{array} \quad \begin{array}{c} \nearrow \psi \\ \text{dashed arrow} \end{array}$$

commute.

This lemma follows from Corollary 3.3 and Proposition 3.6 straightforward.

Lemma 4.18 implies the first assertion of Proposition 4.17. Indeed, since the image of $(C_{\infty,\chi})_{\Gamma_m}$ in F_m^\times/p^N coincides with $\mathcal{W}_{m,N,\chi}(1)$ by Proposition 4.4, we have

$$(\gamma-1)^{a_\chi} \delta_1 \delta_2 \cdot \bar{\varphi}_{m,N,\chi}(\text{the image of } (C_{\infty,\chi})_{\Gamma_m}) \subseteq \psi(\mathcal{W}_{m,N,\chi}(1)) \subseteq \mathfrak{C}_{0,m,N,\chi}.$$

Next, we prove the second assertion. We take arbitrary elements $\delta'_1 \in \text{ann}_{\Lambda_\chi}(\text{Ker } \varphi)$ and $\delta'_2 \in \text{ann}_{\Lambda_\chi}(\text{Coker } \varphi)$. In particular, we may take some p -powers as δ'_1 and δ'_2 . We shall construct a homomorphism

$$\psi_{\delta'_1, \delta'_2} \in \text{Hom}_{R_{m,N,\chi}}(R_{m,N,\chi}, (E_{\infty,\chi})_{\Gamma_m}/p^N),$$

which can be regarded as “inverse” of $\bar{\varphi}_{m,N,\chi}$ in some sense as follows. For each $x \in R_{m,N,\chi}$, we take $y \in (E_{\infty,\chi})_{\Gamma_m}/p^N$ such that

$$\bar{\varphi}_{m,N,\chi}(y) = (\gamma-1)^{a_\chi/2} \delta'_2 x.$$

Then, we define

$$\psi_{\delta'_1, \delta'_2}(x) := (\gamma-1)^{a_\chi/2} \delta'_1 y \in (E_{\infty,\chi})_{\Gamma_m}/p^N.$$

The definition of $\psi_{\delta'_1, \delta'_2}(x)$ is independent of the choice of y , and $\psi_{\delta'_1, \delta'_2}$ is contained in $\text{Hom}_{R_{m,N,\chi}}(R_{m,N,\chi}, (E_{\infty,\chi})_{\Gamma_m}/p^N)$.

Let $f \in \mathcal{H}_{m,N,\chi}(n)$ be an arbitrary homomorphism. Since $R_{m,N,\chi}$ is an injective $R_{m,N,\chi}$ -module, there exists a homomorphism $\tilde{f}: (F_m^\times/p^N)_\chi \longrightarrow R_{m,N,\chi}$ whose restriction to $\mathcal{W}_{m,N,\chi}(1)$ coincides with f . Then, we have an element $a \in R_{m,N,\chi}$ which makes the following diagram

$$\begin{array}{ccccc} & & \xleftarrow{\psi_{\delta'_1, \delta'_2}} & & R_{m,N,\chi} \\ & \swarrow & & \searrow & \uparrow \\ (E_{\infty,\chi})_{\Gamma_m}/p^N & & (C_{\infty,\chi})_{\Gamma_m}/p^N & & R_{m,N,\chi} \\ & \swarrow & \downarrow & \searrow & \uparrow \\ & & \mathcal{W}_{m,N,\chi}(1) & & \\ & \swarrow & & \searrow & \uparrow \\ (F_m^\times/p^N)_\chi & & & & R_{m,N,\chi} \end{array} \quad \begin{array}{c} \text{dashed arrow } \tilde{f} \\ \text{dotted arrow } \times a \end{array}$$

commute, where the right vertical arrow $\times a$ is the scalar multiplication by a , and maps $i: (C_{\infty, \chi})_{\Gamma_m}/p^N \rightarrow (E_{\infty, \chi})_{\Gamma_m}/p^N$ and $j: (C_{\infty, \chi})_{\Gamma_m}/p^N \rightarrow \mathcal{W}_{m, N, \chi}(1)$ are the natural homomorphism. From this diagram, we obtain

$$(\gamma - 1)^{a_x} \delta'_1 \delta'_2 f(\mathcal{W}_{m, N, \chi}(1))_{\Gamma_m}/p^N = a \bar{\varphi}_{m, N, \chi} \circ i((C_{\infty, \chi})_{\Gamma_m}/p^N).$$

Since the characteristic ideal $\text{char}_{\Lambda_\chi}(E_{\infty, \chi}/C_{\infty, \chi})$ is a principal ideal of Λ_χ not containing neither p nor $\gamma - 1$, the second assertion of this proposition follows. \square

5. KURIHARA'S ELEMENTS FOR CIRCULAR UNITS

In this section, we recall some basic facts on the Euler system of circular units, and define some elements $x_{m, N}(n)_\chi$ of $(F_m^\times/p^N)_\chi$, which are analogue of Kurihara's elements in [Ku].

5.1. Let $\chi \in \widehat{\Delta}$ be a character. Recall we put $R_{m, N} := \mathbb{Z}/p^N[\text{Gal}(F_m/\mathbb{Q})]$ and $R_{m, N, \chi} := R_{m, N} \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\chi$. Here, for each $\ell \in \mathcal{S}_N$, we shall recall the definition of two homomorphisms

$$[\cdot]_{m, N, \chi}^\ell: (F_m^\times/p^N)_\chi \rightarrow R_{m, N, \chi} \quad (\text{cf. Definition 5.1}),$$

$$\bar{\phi}_{m, N, \chi}^\ell: (F_m^\times/p^N)_\chi \rightarrow R_{m, N, \chi} \quad (\text{cf. Definition 5.2}),$$

which are important in the induction part in the Euler system arguments. The homomorphism $[\cdot]_{m, N, \chi}^\ell$ is defined by the valuations of the places above ℓ , and $\bar{\phi}_{m, N, \chi}^\ell$ is defined by the local reciprocity maps.

First, we define $[\cdot]_{m, N, \chi}^\ell$. Let F be an algebraic number field. We define

$$\mathcal{I}_F := \text{Div}(\text{Spec}(\mathcal{O}_F))$$

to be the divisor group, and we write its group law additively. We define the homomorphism $(\cdot)_F: F^\times \rightarrow \mathcal{I}_F$ by

$$(x)_F = \sum_{\lambda} \text{ord}_\lambda(x) \lambda,$$

where λ runs through all prime ideals of \mathcal{O}_F , and ord_λ is the normalized valuation of λ . For any prime number ℓ , we define \mathcal{I}_F^ℓ to be the subgroup of \mathcal{I}_F generated by all prime ideals above ℓ . Then, we define $(\cdot)_F^\ell: F^\times \rightarrow \mathcal{I}_F^\ell$ by

$$(x)_F^\ell = \sum_{\lambda|\ell} \text{ord}_\lambda(x) \lambda.$$

Recall that we fix a family of embeddings $\{\ell_{\overline{\mathbb{Q}}}: \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\}_{\ell: \text{prime}}$ (cf. §1 Notation). We denote the ideal of F corresponding to the embedding $\ell_{\overline{\mathbb{Q}}}|_K$ by ℓ_F for each prime number ℓ and algebraic number field F . Assume F/\mathbb{Q} is Galois extension, and ℓ splits completely in F/\mathbb{Q} . Then, \mathcal{I}_F^ℓ is a free $\mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$ -module generated by ℓ_F , and we identify \mathcal{I}_F^ℓ with $\mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$ by the isomorphism $\iota: \mathbb{Z}[\text{Gal}(F/\mathbb{Q})] \xrightarrow{\simeq} \mathcal{I}_F^\ell$ defined by $x \mapsto x \cdot \ell_F$ for $x \in \mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$. We also denote the composition map $F^\times \rightarrow \mathcal{I}_F^\ell \xrightarrow{\iota^{-1}} \mathbb{Z}[\text{Gal}(F/\mathbb{Q})]$ by $(\cdot)_F^\ell$.

Definition 5.1. We define the $R_{m,N,\chi}$ -homomorphism

$$[\cdot]_{m,N,\chi}: (F_m^\times/p^N)_\chi \longrightarrow (\mathcal{I}_{F_m}/p^N)_\chi$$

to be the homomorphism induced by $(\cdot)_{F_m}^\ell: F_m^\times \longrightarrow \mathcal{I}_{F_m}$. Let $\ell \in \mathcal{S}_N$ be any element. We define the $R_{m,N,\chi}$ -homomorphism

$$[\cdot]_{m,N,\chi}^\ell: (F_m^\times/p^N)_\chi \longrightarrow R_{m,N,\chi} = \mathbb{Z}/p^N[\text{Gal}(F_m/\mathbb{Q})] \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\chi$$

to be the homomorphism induced by $(\cdot)_{F_m}^\ell: F_m^\times \longrightarrow \mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})]$.

Second, we will define $\bar{\phi}_{m,N,\chi}^\ell$. Let $\ell \in \mathcal{S}_N$ be any element. Since we assume $N \geq m+1$, the prime number ℓ splits completely in F_m/\mathbb{Q} , and we have $F_{m,\lambda} = \mathbb{Q}_\ell$ for any prime ideal λ of F_m above ℓ . We regard the groups $\bigoplus_{\lambda|\ell} F_{m,\lambda}^\times$ and $\bigoplus_{\lambda|\ell} H_\ell$ are regarded as $\mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})]$ -modules by the identification

$$\begin{aligned} \bigoplus_{\lambda|\ell} F_{m,\lambda}^\times &= \mathcal{I}_{F_m}^\ell \otimes_{\mathbb{Z}} \mathbb{Q}_\ell^\times, \\ \bigoplus_{\lambda|\ell} H_\ell &= \mathcal{I}_{F_m}^\ell \otimes H_\ell, \end{aligned}$$

respectively. Here, we regard \mathbb{Q}_ℓ^\times as a $\mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})]$ -modules on which $\text{Gal}(F_m/\mathbb{Q})$ acts trivially. We denote by

$$\phi_{\mathbb{Q}_\ell}: \mathbb{Q}_\ell^\times \longrightarrow \text{Gal}(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell) = \text{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q}) = H_\ell$$

the reciprocity map of local class field theory defined by

$$\phi_{\mathbb{Q}_\ell}(\ell) = (\ell_{\mathbb{Q}(\mu_\ell)}, \mathbb{Q}(\mu_\ell)/\mathbb{Q}).$$

In this paper, we define the homomorphism

$$\phi_m^\ell: F_m^\times \longrightarrow \mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})] \otimes H_\ell$$

to be the composition of the three homomorphisms of $\mathbb{Z}[\text{Gal}(F_m(\mu_n)/\mathbb{Q})]$ -modules:

$$\begin{aligned} \text{diag}: F_m^\times &\longrightarrow \bigoplus_{\lambda|\ell} F_{m,\lambda}^\times, \\ \oplus \phi_{\mathbb{Q}_\ell}: \bigoplus_{\lambda|\ell} F_{m,\lambda}^\times &\longrightarrow \bigoplus_{\lambda|\ell} H_\ell, \\ \iota_H^{-1}: \bigoplus_{\lambda|\ell} H_\ell &\xrightarrow{\simeq} \mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})] \otimes H_\ell, \end{aligned}$$

which are defined as follows:

- (1) the first homomorphism diag is the diagonal inclusion;
- (2) the second homomorphism $\oplus \phi_{\mathbb{Q}_\ell}$ is the direct sum of the reciprocity maps;
- (3) the third isomorphism ι_H^{-1} is the inverse of the isomorphism

$$\iota_H: \mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})] \otimes H_\ell \xrightarrow{\simeq} \bigoplus_{\lambda|\ell} H_\ell = \mathcal{I}_{F_m}^\ell \otimes H_\ell,$$

which is induced by the above isomorphism $\iota: \mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})] \xrightarrow{\simeq} \mathcal{I}_{F_m}^\ell$ given by $x \mapsto x \cdot \ell_{F_m}$.

Definition 5.2. Let $\ell \in \mathcal{S}_N$ be any element. We define

$$\phi_{m,N,\chi}^\ell: (F_m^\times/p^N)_\chi \longrightarrow \mathbb{Z}/p^N[\text{Gal}(F_m/\mathbb{Q})]_\chi \otimes H_\ell$$

to be the homomorphism of $R_{m,N,\chi}$ -modules induced by ϕ_m^ℓ . The choice of a generator σ_ℓ of H_ℓ induces the $R_{m,N,\chi}$ -homomorphism

$$\bar{\phi}_{m,N,\chi}^\ell: (F_m^\times/p^N)_\chi \longrightarrow \mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})]_\chi = R_{m,N,\chi}.$$

The following formulas on Kolyvagin derivatives are well-known.

Proposition 5.3. *Let n be an integer contained in \mathcal{N}_N . Let $d \in \mathbb{Z}_{>1}$ be an integer dividing \mathfrak{f}_K and $a \in \mathbb{Z}$ an integer coprime to p . For simplicity, we denote $\kappa_{m,N}^d(n)$ or $\kappa_{m,N}^{1,a}(n)$ by $\kappa^\bullet(n)$.*

- (1) *If λ is a prime ideal of K not dividing n , the λ -component of $[\kappa^\bullet(n)_\chi]_{m,N,\chi}$ is 0. In particular, if $q \in \mathcal{S}_N$ is a prime number not dividing n , we have*

$$[\kappa^\bullet(n)_\chi]_{m,N,\chi}^q = 0.$$

(See [Grei] Lemma 3.6 and [Ru2] Proposition 2.4.)

- (2) *Let ℓ be a prime number dividing n . Then,*

$$[\kappa^\bullet(n)_\chi]_{m,N,\chi}^\ell = \bar{\phi}_{m,N,\chi}^\ell(\kappa^\bullet(n/\ell)_\chi).$$

(See [Grei] Lemma 3.6 and [Ru2] Proposition 2.4.)

- (3) *If n is well-ordered, then*

$$\bar{\phi}_{m,N,\chi}^\ell(\kappa_{m,N}^\bullet(n)_\chi) = 0$$

for each prime number ℓ dividing n . (See [MR] Theorem A.4.)

5.2. In this subsection, we will define the Kurihara's elements $x_{\nu,q} \in (F_m^\times/p^N)_\chi$ which become a key of the proof of Theorem 7.1.

We fix circular units

$$\eta_m(n) := \prod_{d|\mathfrak{f}_K} \eta_m^d(n)^{u_d} \times \prod_{i=1}^r \eta_m^{1,a_i}(n)^{v_i} \in F_m^\times$$

for any $m \in \mathbb{Z}_{\geq 0}$ and any $n \in \mathbb{Z}_{\geq 1}$ with $(n, p\mathfrak{f}_K) = 1$, where $r \in \mathbb{Z}_{>0}$, u_d and v_i are elements of $\mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})]$ for each positive integers d and i with $d|\mathfrak{f}_K$ and $1 \leq i \leq r$, and a_1, \dots, a_r are integers prime to p . Here, we assume that r, a_1, \dots, a_r, u_d 's and v_i 's are constant independent of m and n . Then, we put

$$\kappa_{m,N}(\eta; n) := \prod_{d|\mathfrak{f}_K} \kappa_{m,N}^d(n)^{u_d} \times \prod_{i=1}^r \kappa_{m,N}^{a_i}(n)^{v_i} \in F_m^\times/p^N.$$

Note that $\kappa_{m,N}(\eta, \chi)$ is the Kolyvagin derivative of the Euler system

$$\eta = \{\eta_m(n)_\chi \in (F_m(\mu_n)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi\}_{m,n}$$

of circular units. If no confusion arises, we write $\kappa(n) = \kappa_{m,N}(\eta; n)$ for simplicity.

Definition 5.4. Let $q\nu = q \prod_{i=1}^r \ell_i \in \mathcal{N}_N$, where q, ℓ_1, \dots, ℓ_r are distinct prime numbers. For any $e \in \mathbb{Z}_{>0}$ dividing ν , we define $\tilde{\kappa}_{\{e,q\}} \in (F_m^\times/p^N)_\chi \otimes (\bigotimes_{\ell|d} H_\ell)$ by

$$\tilde{\kappa}_{\{e,q\}} := \kappa(qe)_\chi \otimes \left(\bigotimes_{\ell|e} \sigma_\ell \right).$$

Let $q\nu \in \mathcal{N}_N$ and assume $q\nu$ is *well-ordered*. Assume that for each prime number ℓ dividing ν , an element $w_\ell \in R_{m,N,\chi} \otimes H_\ell$ is given. Then, we have an element $\bar{w}_\ell \in R_{m,N,\chi}$ such that $w_\ell = \bar{w}_\ell \otimes \sigma_\ell$. Note that we will take $\{w_\ell\}_{\ell|\nu}$ explicitly later, but here, we take arbitrary one. For any $e \in \mathbb{Z}_{>0}$ dividing ν , we define

$$w_e := \bigotimes_{\ell|e} w_\ell \in R_{m,N,\chi} \otimes \left(\bigotimes_{\ell|e} H_\ell \right).$$

We also define the element $\bar{w}_e \in R_{m,N,\chi}$ by $w_e = \bar{w}_e \otimes \left(\bigotimes_{\ell|e} \sigma_\ell \right)$.

Note that we write the group law of $(F_m^\times/p^N)_\chi \otimes \left(\bigotimes_{\ell|e} H_\ell \right)$ multiplicatively.

Definition 5.5. We define the element $\tilde{x}_{\nu,q}$ by

$$\tilde{x}_{\nu,q} := \prod_{e|\nu} w_e \otimes \tilde{\kappa}_{\{\nu/e,q\}} \in (F_m^\times/p^N)_\chi \otimes \left(\bigotimes_{\ell|e} H_\ell \right).$$

Note that we naturally identify the $R_{m,N,\chi}$ -module $(F_m^\times/p^N)_\chi \otimes \left(\bigotimes_{\ell|e} H_\ell \right)$ with

$$R_{m,N,\chi} \otimes \left(\bigotimes_{\ell|e} H_\ell \right) \otimes_{R_{m,N,\chi}} (F_m^\times/p^N)_\chi.$$

The element $x_{\nu,q} \in (F_m^\times/p^N)_\chi$ is defined by $\tilde{x}_{\nu,q} = x_{\nu,q} \otimes \left(\bigotimes_{\ell|\nu} \sigma_\ell \right)$.

The following formulas follows from Proposition 5.3 straightforward.

Proposition 5.6 (cf. [Ku] Proposition 6.1). *Let $q\nu \in \mathcal{N}_N$ and we assume that $q\nu$ is well-ordered.*

- (1) *If λ is a prime ideal of K not dividing n , the λ -component of $[x_{\nu,q}]_{m,N,\chi}$ is 0. In particular, if s is a prime number not dividing $q\nu$, we have*

$$[x_{\nu,q}]_{m,N,\chi}^s = 0.$$

- (2) *Let ℓ be a prime number dividing ν . Then, we have*

$$[x_{\nu,q}]_{m,N,\chi}^\ell = \bar{\phi}_{m,N,\chi}^\ell(x_{\nu/\ell,q}).$$

- (3) *Let ℓ be a prime number dividing ν . Then, we have*

$$\bar{\phi}_{m,N,\chi}^\ell(x_{\nu,q}) = \bar{w}_\ell \bar{\phi}_{m,N,\chi}^\ell(x_{\nu/\ell,q}).$$

6. CHEBOTAREV DENSITY THEOREM

Recall that we fix a family of embeddings $\{\ell_{\overline{\mathbb{Q}}}: \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\}_{\ell:\text{prime}}$ satisfying the technical condition (Chb) for families of embeddings as follows:

(Chb) For any subfield $F \subset \overline{\mathbb{Q}}$ which is a finite Galois extension of \mathbb{Q} and any element $\sigma \in \text{Gal}(F/\mathbb{Q})$, there exist infinitely many prime numbers ℓ such that ℓ is unramified in F/\mathbb{Q} and $(\ell_F, F/\mathbb{Q}) = \sigma$, where ℓ_F is the prime ideal of F corresponding to the embedding $\ell_{\overline{\mathbb{Q}}}|_F$.

Note that the existence of such a family of embeddings follows from the Chebotarev density theorem. We need the condition (Chb) in the proof of Proposition 6.1.

Let $\chi \in \widehat{\Delta}$ be any element. Recall that we denote the restriction of χ to Δ_0 by χ_0 , and we put

$$a_\chi := \begin{cases} 0 & \text{if } \chi(p) \neq 1; \\ 2 & \text{if } \chi(p) = 1. \end{cases}$$

Here, we shall prove Proposition 6.1, which is the key of induction argument in the proof of Theorem 7.1. This proposition corresponds to Lemma 9.1 in [Ku].

Proposition 6.1. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. Assume $qn = q \prod_{i=1}^r \ell_i \in \mathcal{N}_N$, where q, ℓ_1, \dots, ℓ_r are prime numbers. Suppose the following are given:*

- a finite $R_{m,N,\chi}$ -submodule W of $(F_m^\times/p^N)_\chi$;
- a $R_{m,N,\chi}$ -homomorphism $\psi: W \longrightarrow R_{m,N,\chi}$.

Then, there exist infinitely many $q' \in \mathcal{S}_N$ which splits completely in $F_m(\mu_n)/\mathbb{Q}$, and satisfies all of the following properties.

- (1) *the class of q'_{F_m} in $A_{m,\chi}$ coincides with the class of $|\Delta_p| \cdot q_{F_m}$. (Recall that we write the group law of \mathcal{I}_F additively.)*
- (2) *there exists an element $z \in (F_m^\times \otimes \mathbb{Z}_p)_\chi$ such that*
 - $(z)_{m,\chi} = (q'_{F_m} - |\Delta_p| \cdot q_{F_m})_\chi \in (\mathcal{I}_{F_m} \otimes \mathbb{Z}_p)_\chi$,
 - $\phi_{m,N,\chi}^{\ell_i}(z) = 0$ for each $i = 1, \dots, r$.
- (3) *the group W is contained in the kernel of $[\cdot]_{m,N,\chi}^{q'}$, and*

$$\psi(x) = |\Delta_p|^2 \bar{\phi}_{m,N,\chi}^{q'}(x)$$

for any $x \in W$.

Proof. Proof of this proposition is essentially the same as those of Lemma 9.1 in [Ku] though we have to treat carefully when $|\Delta_p| \neq 1$. We shall prove this proposition in four steps.

The first step. Let v be a prime ideal of F_m . We denote the ring of integers of the completion $F_{m,v}$ of F_m at v by $\mathcal{O}_{F_{m,v}}$, and define the subgroup $\mathcal{O}_{F_{m,v}}^1$ of $\mathcal{O}_{F_{m,v}}^\times$ by

$$\mathcal{O}_{F_{m,v}}^1 := \{x \mid x \equiv 1 \pmod{\mathfrak{m}_v}\},$$

where \mathfrak{v} is the maximal ideal of $\mathcal{O}_{F_{m,v}}$. We denote the residue field of F_m at v by $k(v)$. Let $F_m\{n\}$ be the maximal abelian p -extension of F_m unramified outside n . By

global class field theory, we have the isomorphism

$$\frac{(\prod_{v|n} F_{m,v}^\times / \mathcal{O}_{F_{m,v}}^1) \times (\bigoplus_{u \nmid n} F_{m,u}^\times / \mathcal{O}_{F_{m,u}}^\times)}{F_m^\times} \otimes \mathbb{Z}_p \xrightarrow{\simeq} \text{Gal}(F_m\{n\}/F_m),$$

where u runs all finite places outside n . This isomorphism induces the homomorphism

$$\iota: \bigoplus_{v|n} k(v)^\times \otimes \mathbb{Z}_p \longrightarrow \text{Gal}(F_m\{n\}/F_m).$$

Taking the χ -quotients, we obtain the homomorphism

$$\iota_\chi: \left(\bigoplus_{v|n} k(v)^\times \otimes \mathbb{Z}_p \right)_\chi \longrightarrow \text{Gal}(F_m\{n\}/F_m)_\chi$$

of $\mathbb{Z}_p[\text{Gal}(F_m/\mathbb{Q})]_\chi$ -modules. We denote by $F_m\{n\}_\chi$ the intermediate field of $F_m\{n\}/F_m$ with $\text{Gal}(F_m\{n\}_\chi/F_m) = \text{Gal}(F_m\{n\}/F_m)_\chi$. Recall $n = \prod_{i=1}^r \ell_i$, and all prime divisors ℓ_i of n split completely in F_m/\mathbb{Q} . By local Artin maps, we obtain the isomorphism

$$\left(\bigoplus_{v|n} k(v)^\times \otimes \mathbb{Z}_p \right)_\chi \xrightarrow{\simeq} \bigoplus_{i=1}^r (\mathbb{Z}_p[\text{Gal}(F_m/\mathbb{Q})]_\chi \cdot \ell_{i,F_m}) \otimes H_{\ell_i},$$

and we identify them by this isomorphism.

Let $L := F_m\{n\}_\chi K(\mu_{np^N})$ be the composition field. Note that the cokernel of the natural homomorphism

$$\text{Gal}(L/F_m) \longrightarrow \text{Gal}(L/F_m)_\chi \times \text{Gal}(L/F_m)_1$$

is annihilated by $|\Delta_p|$ since $\chi \neq 1$. Since the subgroup Δ of $\text{Gal}(F_m/\mathbb{Q})$ acts on $\text{Gal}(F_m\{n\}_\chi/F_m)$ via χ , $\text{Gal}(F_m\{n\}_\chi/F_m)$ is a quotient of $\text{Gal}(L/F_m)_\chi$. On the other hand, since Δ of $\text{Gal}(F_m/\mathbb{Q})$ acts on $\text{Gal}(K(\mu_{np^N})/F_m)$ via the trivial character, $\text{Gal}(K(\mu_{np^N})/F_m)$ is a quotient of $\text{Gal}(L/F_m)_1$. Then, the cokernel of the natural homomorphism

$$\text{Gal}(L/F_m) \longrightarrow \text{Gal}(F_m\{n\}_\chi/F_m) \times \text{Gal}(K(\mu_{np^N})/F_m)$$

is annihilated by $|\Delta_p|$. Then, we take an element $\sigma \in \text{Gal}(L/K(\mu_{np^N}))$ such that

$$\sigma|_{F_m\{n\}_\chi} = (q_{F_m\{n\}_\chi}, F_m\{n\}_\chi/F_m)^{|\Delta_p|}.$$

The second step. We fix a finite $R_{m,\chi}$ -submodule \mathcal{W} of F_m^\times/p^N whose image in $(F_m^\times/p^N)_\chi$ is W . By Corollary 3.3, we take a homomorphism $\tilde{\psi} \in \text{Hom}_{R_{m,N}}(\mathcal{W}, R_{m,N})$ which makes the diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\tilde{\psi}} & R_{m,N} \\ \downarrow & & \downarrow \\ W & \xrightarrow{|\Delta_p|^2 \psi} & R_{m,N,\chi} \end{array}$$

commute. We define a projection $\text{pr}: R_{m,N} \longrightarrow \mathbb{Z}/p^N\mathbb{Z}$ by

$$\sum_{g \in \text{Gal}(F_m/\mathbb{Q})} a_g g \longmapsto a_1,$$

where $a_g \in \mathbb{Z}/p^N\mathbb{Z}$ for all $g \in \text{Gal}(F_m/\mathbb{Q})$, and $1 \in \text{Gal}(F_m/\mathbb{Q})$ is the unit. This projection induces an *isomorphism*

$$P: \text{Hom}_{R_{m,N}}(\mathcal{W}, R_{m,N}) \longrightarrow \text{Hom}(\mathcal{W}, \mathbb{Z}/p^N\mathbb{Z})$$

given by $f \mapsto \text{pr} \circ f$. We define a homomorphism

$$(P\tilde{\psi})_1: \mathcal{W} \longrightarrow \mu_{p^N}$$

of abelian groups by $x \mapsto \zeta_{p^N}^{P(\tilde{\psi})(x)}$.

We denote the image of \mathcal{W} in $K(\mu_{p^N})^\times/p^N$ by \mathcal{W}' . Let M be the extension field of $K(\mu_{p^N})$ generated by all p^N -th roots of elements of $K(\mu_{p^N})^\times$ whose image in $K(\mu_{p^N})^\times/p^N$ is contained in \mathcal{W}' . So, the Kummer pairing induces the isomorphism

$$\text{Kum}: \text{Gal}(M/K(\mu_{p^N})) \xrightarrow{\cong} \text{Hom}(\mathcal{W}', \mu_{p^N}).$$

Note that the complex conjugation c acts on $H^1(K(\mu_{p^N})/F_m, \mu_{p^N})$ by -1 , and c acts on F_m^\times/p^N trivially. So, the group $H^1(K(\mu_{p^N})/F_m, \mu_{p^N})$ vanishes, and the natural homomorphism $F_m^\times/p^N \rightarrow K(\mu_{p^N})^\times/p^N$ is an injection. This implies that the natural homomorphism

$$i: \mathcal{W} \longrightarrow \mathcal{W}'$$

is an isomorphism. We take $\lambda \in \text{Gal}(M/K(\mu_{p^N}))$ such that

$$i^* \circ \text{Kum}(\lambda) = (P\tilde{\psi})_1.$$

The third step. Recall that $LM/K(\mu_{p^N})$ is an abelian p -extension, so we regard $\text{Gal}(LM/K(\mu_{p^N}))$ as a $\mathbb{Z}_p[\text{Gal}(K(\mu_{p^N})/\mathbb{Q})]$ -module. We have the natural isomorphism:

$$\text{Gal}(LM/K(\mu_{p^N})) \hookrightarrow \text{Gal}(LM/K(\mu_{p^N}))_+ \times \text{Gal}(LM/K(\mu_{p^N}))_- ,$$

where $\text{Gal}(LM/K(\mu_{p^N}))_+$ (resp. $\text{Gal}(LM/K(\mu_{p^N}))_-$) denotes the maximal quotient of $\text{Gal}(LM/K(\mu_{p^N}))$ on which the complex conjugation c acts trivially (resp. by -1). We put $\tilde{\Delta} := \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$, and regard $\tilde{\Delta}$ as a subgroup of $\text{Gal}(\mathbb{Q}(\mu_{p^N})/\mathbb{Q})$. Note that on the one hand, $\text{Gal}(L/K(\mu_{p^N}))$ is a quotient of $\text{Gal}(LM/K(\mu_{p^N}))_+$ since $\tilde{\Delta}$ acts trivially on $\text{Gal}(L/K(\mu_{p^N}))$. On the other hand, The complex conjugation c acts on $\text{Gal}(M/K(\mu_{p^N}))$ by -1 since $\tilde{\Delta}$ acts on $\text{Gal}(M/K(\mu_{p^N}))$ via the character $\chi^{-1}\omega$. This implies $\text{Gal}(M/K(\mu_{p^N}))$ is a quotient of $\text{Gal}(LM/K(\mu_{p^N}))_-$. Therefore, the natural homomorphism

$$\text{Gal}(LM/K(\mu_{p^N})) \hookrightarrow \text{Gal}(L/K(\mu_{p^N})) \times \text{Gal}(M/K(\mu_{p^N}))$$

is a surjection. By the condition (Chb), there exists infinitely many prime numbers q' such that

$$\begin{cases} (q'_L, L/K(\mu_{p^N})) = \sigma \\ (q'_M, M/K(\mu_{p^N})) = \lambda^{-1}. \end{cases}$$

The fourth step. Here, we prove that each of such q' unramified in L/\mathbb{Q} satisfies conditions (1)-(3) of Proposition 6.1. First, we show q' satisfies conditions (1) and (2). Let $\alpha = (\alpha_v)_v \in \mathbb{A}_{F_m}^\times$ be an idele whose q'_{F_m} -component is a uniformizer of $F_{m,q'_{F_m}}$, and other components are 1. Let $\beta = (\beta_v)_v \in \mathbb{A}_{F_m}^\times$ be an element as follows. The components above q are given by

$$|\Delta_p| \cdot (x_v)_{v|q} \in \prod_{v|q} F_{m,v}^\times,$$

where $x_{q_{F_m}}$ is a uniformizer of $F_{m,q_{F_m}}$, and $x_v = 1$ otherwise. For all places v of F_m not above q , we put $\beta_v = 1$.

By definition, ideles α and β have the same image in the group

$$\left(\frac{(\prod_{v|n} F_{m,v}^\times / \mathcal{O}_{F_m,v}^1) \times (\bigoplus_{u \nmid n} F_{m,u}^\times / \mathcal{O}_{F_m,u}^\times)}{F_m^\times} \otimes \mathbb{Z}_p \right)_\chi \simeq \text{Gal}(F_m\{n\}_\chi / F_m).$$

This implies there exist $z \in (F_m^\times \otimes \mathbb{Z}_p)_\chi$ such that

$$\alpha = z\beta \quad \text{in} \quad \left(\left(\prod_{v|n} F_{m,v}^\times / \mathcal{O}_{F_m,v}^1 \right) \times \left(\bigoplus_{u \nmid n} F_{m,u}^\times / \mathcal{O}_{F_m,u}^\times \right) \otimes \mathbb{Z}_p \right)_\chi.$$

Hence, we have $(z)_{F_m,\chi} = (q'_{F_m} - |\Delta_p| \cdot q_{F_m})_\chi$, and $\phi_{m,N,\chi}^{\ell_i}(z) = 0$ for each $i = 1, \dots, r$. Obviously, the prime number q' satisfies conditions (1) and (2).

Next, we shall prove q' satisfies condition (3). Recall that we have

$$(q'_M, M/K(\mu_{p^N})) = \lambda^{-1}.$$

So, by definition of λ , we have

$$\zeta_{p^N}^{P(\tilde{\psi})(x)} = (x^{1/p^N})^{1 - \text{Fr}_{q'}},$$

for any $x \in W$, where we put

$$\text{Fr}_{q'} := (q'_M, M/K(\mu_{p^N})) \in \text{Gal}(M/K(\mu_{p^N})),$$

and $x^{1/p^N} \in L$ is a p^N -th root of x . Since q' is unramified in M/\mathbb{Q} , the group W is contained in the kernel of $[\cdot]_{m,N,\chi}^{q'}$. So, we obtain

$$\zeta_{p^N}^{P(\tilde{\psi})(x)} \equiv x^{(q'-1)/p^N} \pmod{q'}.$$

We can take the unique intermediate field F of $F_m(\mu_{q'})/F_m$ whose degree over F_m is p^N since $q' \equiv 1 \pmod{p^N}$. We denote the image of $\sigma_{q'} \in H_{q'}$ in

$$H_{q'} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \text{Gal}(F/F_m)$$

by $\bar{\sigma}_{q'}$. Let π be a uniformizer of $F_{q'_F}$. By definition of $\sigma_{q'}$, we have

$$\pi^{\bar{\sigma}_{q'} - 1} \equiv \zeta_{p^N} \pmod{\mathfrak{m}_{q'}},$$

where $\mathfrak{m}_{q'}$ is the maximal ideal of $F_{q'_F}$. Recall that W is contained in the kernel of $[\cdot]_{m,N,\chi}^{q'}$. We put

$$\phi(x) := \bar{\sigma}_{q'}^{P(\bar{\phi}_{m,N}^{q'})(x)} \in \text{Gal}(F/F_m).$$

By [Se] Chapter XIV Proposition 6, we have

$$\zeta_{p^N}^{P(\bar{\phi}_{m,N}^{q'})(x)} = \pi^{\phi(x)-1} \equiv x^{(1-q')/p^N} \pmod{\mathfrak{m}_{q'}}$$

for all $x \in W$. Hence, we obtain

$$\zeta_{p^N}^{P(\tilde{\psi})(x)} = \zeta_{p^N}^{P(\bar{\phi}_{m,N}^{q'})(x)}$$

for all $x \in W$. Therefore q' satisfies condition (3) of Proposition 6.1. \square

7. EULER SYSTEM ARGUMENT VIA KURIHARA'S ELEMENTS

In this section, we prove the assertions of Theorem 1.1 on the upper bounds of the higher Fitting ideals by using Kurihara's Euler system arguments established in [Ku]. Let us state the assertions of Theorem 7.1, which is the goal of this section. Let the ideals $I_{P_\chi^E}$ and $J_{P_\chi^E}$ of Λ_χ be as in Proposition 3.6, the ideal I_C as in §4.3, and the Λ_χ -submodule Y of X_χ as in Proposition 3.9. We denote the ideal of Λ_χ generated by i -th power of elements of $\text{ann}_{\Lambda_\chi}(Y/(\gamma-1)X)_\chi$ by I_i for each $i \in \mathbb{Z}_{\geq 0}$. The goal of this section is the following theorem.

Theorem 7.1. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. Then, we have*

- (i) $\mathfrak{C}_{0,\chi} \subseteq \text{Fitt}_{\Lambda_\chi,0}(X'_\chi)$.
- (ii-0) $(\gamma-1)^{a_\chi} |\Delta_p|^4 I_C I_{P_\chi^E} J_{P_\chi^E} \text{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$.
- (ii-i) $(\gamma-1)^{a_\chi} |\Delta_p|^{6+4i} I_i I_C I_{P_\chi^E} J_{P_\chi^E} \text{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$ for any $i \in \mathbb{Z}_{\geq 1}$.

In particular, we have

- (i) $\mathfrak{C}_{0,\chi} \prec \text{Fitt}_{\Lambda_\chi,0}(X_\chi)$.
- (ii-0) $(\gamma-1)^{a_\chi} |\Delta_p|^4 \text{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$.
- (ii-i) $(\gamma-1)^{a_\chi} |\Delta_p|^{6+4i} \text{Fitt}_{\Lambda_\chi,i}(X_\chi) \prec \mathfrak{C}_{i,\chi}$ for any $i \in \mathbb{Z}_{\geq 1}$.

The assertions of Theorem 1.1 on the upper bounds of the higher Fitting ideals are special cases of Theorem 7.1.

Corollary 7.2. *Assume that the extension degree of K/\mathbb{Q} is prime to p , and $\chi \in \widehat{\Delta}$ is a character satisfying $\chi(p) \neq 1$. Then, we have the following.*

- (i) $\mathfrak{C}_{0,\chi} \subseteq \text{Fitt}_{\Lambda_\chi,0}(X'_\chi)$.
- (ii) $\text{ann}_{\Lambda_\chi}(X_{\chi,\text{fin}}) \text{Fitt}_{\Lambda_\chi,i}(X_\chi) \subseteq \mathfrak{C}_{i,\chi}$ for any $i \in \mathbb{Z}_{\geq 0}$.

Proof. Here, let us deduce the corollary from Theorem 7.1. Under the assumption of Corollary 7.2, we have $|\Delta_p| = 1$, and we can take $I_i = I_C = I_{P_\chi^E} = \Lambda_\chi$ and $J_{P_\chi^E} = \text{ann}_{\Lambda_\chi}(X_{\chi,\text{fin}})$ by Proposition 3.7, Proposition 3.10 and Proposition 4.5. The corollary follows from these results. \square

7.1. We spend this subsection on the setting of notations. We assume that $\chi \in \widehat{\Delta}$ is non-trivial. Let \mathfrak{m}_χ be the maximal ideal of Λ_χ . Recall that we denote by $X_{\chi, \text{fin}}$ the maximal pseudo-null submodule of X_χ , and put $X'_\chi := X_\chi / X_{\chi, \text{fin}}$. Since X'_χ has no non-trivial pseudo-null submodules, we have an exact sequence

$$(5) \quad 0 \longrightarrow \Lambda_\chi^h \xrightarrow{f} \Lambda_\chi^h \xrightarrow{g} X'_\chi \longrightarrow 0,$$

by Lemma 2.7. Let M be the matrix corresponding to f with respect to the standard basis $(\mathbf{e}_i)_{i=1}^h$ of Λ_χ^h . We may assume that all entries of M are contained in \mathfrak{m}_χ . In particular, we have

$$(6) \quad \mathbf{e}_i - a\mathbf{e}_j \notin \text{Ker } g$$

for all $i, j \in \mathbb{Z}$ with $1 \leq i \neq j \leq h$ and all $a \in \Lambda_\chi$.

Let $\{m_1, \dots, m_h\}$ and $\{n_1, \dots, n_h\}$ be permutations of $\{1, \dots, h\}$, and let i be an integer satisfying $1 \leq i \leq h-1$. Let us consider the matrix M_i which is obtained from M by eliminating the n_j -th rows ($j = 1, \dots, i$) and the m_k -th columns ($k = 1, \dots, i$). If $\det(M_i) = 0$, this is trivial, so we assume that $\det(M_i) \neq 0$. If necessary, we permute $\{m_1, \dots, m_i\}$, and assume $\det(M_r) \neq 0$ for all integers r satisfying $0 \leq r \leq i$.

We fix a sequence $\{N_m\}_{m \in \mathbb{Z}_{\geq 0}}$ of positive integers satisfying $N_{m+1} > N_m > m+1$ and $p^{N_m} > |A_m|$ for any $m \in \mathbb{Z}_{\geq 0}$. First, we fix a sufficiently large integer m . For simplicity, we put $F := F_m$, $R := \mathbb{Z}_p[\text{Gal}(F_m/\mathbb{Q})]_\chi$, $N := N_m$ and $R_N := R_{m, N, \chi} = \mathbb{Z}/p^N[\text{Gal}(F_m/\mathbb{Q})]_\chi$. From the exact sequence (5), we obtain the exact sequence

$$0 \longrightarrow R^h \xrightarrow{\bar{f}} R^h \xrightarrow{\bar{g}} X'_{\chi, \Gamma_m} \longrightarrow 0,$$

by taking the Γ_m -coinvariants. Let $A_{m, \chi, \text{fin}}$ be the image of $X_{\chi, \text{fin}}$ in $A_{m, \chi}$ by the natural homomorphism

$$X_\chi \twoheadrightarrow X_{\Gamma_m, \chi} \twoheadrightarrow A_m.$$

We put $A'_{m, \chi} := A_{m, \chi} / A_{\text{fin}, \chi}$. The image of \mathbf{e}_r in R^h is denoted by $\mathbf{e}_r^{(m)}$. For each $i \in \mathbb{Z}$ with $1 \leq i \leq h$, we denote by $\mathbf{c}_r^{(m)} \in A'_{m, \chi}$ the image of $\mathbf{e}_r^{(m)}$ by the homomorphism

$$R^h \xrightarrow{\bar{g}} X'_{\chi, \Gamma_m} \twoheadrightarrow A'_m.$$

We fix a lift $\tilde{\mathbf{c}}_r^{(m)} \in A_{m, \chi}$ of $\mathbf{c}_r^{(m)}$. The condition (6) and Proposition 3.9 imply that if necessary, we replace m with larger one, and we may assume $\tilde{\mathbf{c}}_r^{(m)} \neq \tilde{\mathbf{c}}_s^{(m)}$ for any $r, s \in \mathbb{Z}$ with $1 \leq r, s \leq h$ and $r \neq s$. If the extension degree of K/\mathbb{Q} is divisible by p , then we additionally assume that $\tilde{\mathbf{c}}_r^{(m)} \neq |\Delta_p| \cdot \tilde{\mathbf{c}}_s^{(m)}$ for any $r, s \in \mathbb{Z}$ satisfying $1 \leq r, s \leq h$. We define

$$\begin{aligned} P_r &:= \{\ell \in \mathcal{S}_N \mid [\ell_F]_\chi = \tilde{\mathbf{c}}_r^{(m)}\}; \\ P'_r &:= \{\ell' \in \mathcal{S}_N \mid [\ell'_F]_\chi = |\Delta_p| \cdot \tilde{\mathbf{c}}_r^{(m)}\}, \end{aligned}$$

where $[\ell_F]_\chi$ is the ideal class of ℓ_F in $A_{m,\chi}$. By the condition (Chb), note that P_r and P'_r are not empty for all r . We define a set P of prime numbers by the union

$$P := \prod_{r=1}^i (P_r \cup P'_r),$$

and we denote by P_F the set of all prime ideals of \mathcal{O}_F above primes contained in P .

Let J be the subgroup of \mathcal{I}_F generated by P_F , and \mathcal{J} the image of $(J \otimes \mathbb{Z}_p)_\chi$ in $(\mathcal{I}_F \otimes \mathbb{Z}_p)_\chi$. We denote by \mathcal{F} the inverse image of \mathcal{J} by the homomorphism

$$(\cdot)_F: (F^\times \otimes \mathbb{Z}_p)_\chi \longrightarrow (\mathcal{I}_F \otimes \mathbb{Z}_p)_\chi.$$

We define a surjective homomorphism

$$\alpha: \mathcal{J} \longrightarrow R^h$$

by $\ell_F \mapsto \mathbf{e}_r$ (resp. $\ell'_F \mapsto |\Delta_p| \cdot \mathbf{e}_r$) for each $\ell \in P_r$ (resp. $\ell \in P'_r$) and r with $1 \leq r \leq h$. We define

$$\alpha_r := \text{pr}_r \circ \alpha: \mathcal{J} \xrightarrow{\alpha} R^h \xrightarrow{\text{pr}_r} R$$

to be the composition of α and the r -th projection. We consider the following diagram

$$\begin{array}{ccccccc} \mathcal{F} & \xrightarrow{(\cdot)_{F,\chi}} & \mathcal{J} & \longrightarrow & A'_{m,\chi} & & \\ & & \downarrow \alpha & & \uparrow \iota_A & & \\ 0 \longrightarrow & R^h & \xrightarrow{\bar{f}} & R^h & \xrightarrow{\bar{g}} & X'_{\chi,\Gamma_m} & \longrightarrow 0, \end{array}$$

where ι_A is induced by the canonical homomorphism. We fix a non-zero element $\varepsilon \in \text{ann}_{\Lambda_\chi}(Y/(\gamma-1)X)$. By Lemma 3.9, we can define the homomorphism $\beta: \mathcal{F} \longrightarrow R^h$ to make the diagram

$$(7) \quad \begin{array}{ccccccc} \mathcal{F} & \xrightarrow{(\cdot)_{F,\chi}} & \mathcal{J} & \xrightarrow{\varepsilon \cdot \pi'_A} & A'_{m,\chi} & & \\ \downarrow \beta & & \downarrow \varepsilon \cdot \alpha & & \uparrow \iota_A & & \\ 0 \longrightarrow & R^h & \xrightarrow{\bar{f}} & R^h & \xrightarrow{\bar{g}} & X'_{\chi,\Gamma_m} & \longrightarrow 0 \end{array}$$

commute, where π'_A is the natural homomorphism. Note that since the second row of the diagram is exact, β is well-defined. We define

$$\beta_r := \text{pr}_r \circ \beta: \mathcal{F} \xrightarrow{\beta} R^h \xrightarrow{\text{pr}_r} R$$

to be the composition of β and the r -th projection. We consider the diagram (7) by taking $(- \otimes \mathbb{Z}/p\mathbb{Z})$.

We regard $(F^\times/p^N)_\chi$ as a Λ_χ -module. For an element $x \in (F^\times/p^N)_\chi$ and $\delta \in \text{ann}_{\Lambda_\chi}(X_{\chi,\text{fin}})$, we denote the scalar multiple of x by $\delta \in \Lambda_\chi$ by x^δ .

We need the following two lemmas, namely Lemma 7.3 and 7.4.

Lemma 7.3. *The kernel of the natural homomorphism*

$$\iota_{\mathcal{F},N}: \mathcal{F}/p^N \longrightarrow (F^\times/p^N)_\chi$$

is annihilated by $|\Delta_p|$.

Proof. Let x be an element in the kernel of the homomorphism

$$\iota_{\mathcal{F},N}: \mathcal{F}/p^N \longrightarrow (F^\times/p^N)_\chi$$

and \tilde{x} a lift of x in \mathcal{F} . Then, there exists $y \in F^\times \otimes \mathbb{Z}_p$ such that $\tilde{x} = y_\chi^{p^N}$. Note that all \mathbb{Z}_p -torsion elements of $(\mathcal{I}_F \otimes \mathbb{Z}_p)_\chi / \mathcal{J}$ are annihilated by $|\Delta_p|$ by Corollary 3.2 since the \mathbb{Z}_p -module $(\mathcal{I}_F \otimes \mathbb{Z}_p)/(J \otimes \mathbb{Z}_p)$ is torsion free. Since $(\tilde{x})_{F,\chi} \in \mathcal{J}$, we have $(y^{|\Delta_p|})_{F,\chi} \in \mathcal{J}$. Therefore we have $y^{|\Delta_p|} \in \mathcal{F}$, and we obtain $x^{|\Delta_p|} = 1$. \square

We denote the image of the natural homomorphism $\iota_{\mathcal{F},N}: \mathcal{F}/p^N \longrightarrow (F^\times/p^N)_\chi$ by $\bar{\mathcal{F}}_N$. By Lemma 7.3, there exists an R_N -homomorphism

$$\tilde{\beta}_{r,N}: \bar{\mathcal{F}}_N \longrightarrow R_N$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{F}/p^N & \xrightarrow{\iota_{\mathcal{F},N}} & \bar{\mathcal{F}}_N \\ & \searrow |\Delta_p| \cdot \tilde{\beta}_{r,N} & \downarrow \tilde{\beta}_{r,N} \\ & & R_N \end{array}$$

for each integer r with $1 \leq r \leq h$, where $\tilde{\beta}_{r,N}: \mathcal{F}/p^N \longrightarrow R_N$ is the homomorphism induced by β_r .

Lemma 7.4. *Let $[\cdot]_{F,N,\chi}: (F^\times/p^N)_\chi \longrightarrow (\mathcal{I}_F/p^N)_\chi$ be the homomorphism induced by $(\cdot)_F: F^\times \longrightarrow \mathcal{I}_F$. Let x be an element of $(F^\times/p^N)_\chi$ such that $[x]_{F,N,\chi} \in \mathcal{J}/p^N$. Then, for any $\delta \in \text{ann}_{\Lambda_\chi}(X_{\chi,\text{fin}})$, the element $x^{\delta|\Delta_p|^2}$ is contained in $\bar{\mathcal{F}}_N$.*

Proof. Recall the natural exact sequence:

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{I}_F \otimes \mathbb{Z}_p \longrightarrow A_m \longrightarrow 0,$$

where \mathcal{P} is defined by $\mathcal{P} := (F^\times/\mathcal{O}_F^\times) \otimes \mathbb{Z}_p$. By the snake lemma for the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{I}_F \otimes \mathbb{Z}_p & \longrightarrow & A_m \longrightarrow 0 \\ & & \downarrow \times p^N & & \downarrow \times p^N & & \downarrow \times p^N \\ 0 & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{I}_F \otimes \mathbb{Z}_p & \longrightarrow & A_m \longrightarrow 0, \end{array}$$

we obtain the following exact sequence

$$0 \longrightarrow A_m \longrightarrow \mathcal{P}/p^N \longrightarrow \mathcal{I}_F/p^N \longrightarrow A_m \longrightarrow 0.$$

(Recall we take $p^{N_m} > |A_m|$.)

Let B_m be the image of $J \otimes \mathbb{Z}_p$ in A_m , and $\mathcal{P}_0 \subset \mathcal{P}$ the inverse image of $J \otimes \mathbb{Z}_p$. Then, we have the exact sequence

$$0 \longrightarrow \mathcal{P}_0 \longrightarrow J \otimes \mathbb{Z}_p \longrightarrow B_m \longrightarrow 0,$$

and by a similar argument as above, we obtain the exact sequence

$$0 \longrightarrow B_m \longrightarrow \mathcal{P}_0/p^N \longrightarrow J/p^N \longrightarrow B_m \longrightarrow 0.$$

Now, we obtain the commutative diagram

$$(8) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & B_m & \longrightarrow & \mathcal{P}_0/p^N & \longrightarrow & J/p^N & \longrightarrow & B_m & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_m & \longrightarrow & \mathcal{P}/p^N & \longrightarrow & \mathcal{I}_F/p^N & \longrightarrow & A_m & \longrightarrow & 0 \end{array}$$

whose two rows are exact, and the vertical arrows are injective. Let $\tilde{\delta}$ be an arbitrary element of $\text{ann}_\Lambda(A_m/B_m)$. Let x be an arbitrary element of \mathcal{P}/p^N satisfying $[x]_N \in J/p^N$. Let us show that x^δ is contained in \mathcal{P}_0/p^N . By the diagram (8), there exists an element $y \in \mathcal{P}_0/p^N$ satisfying $[x]_N = [y]_N$. Since $[xy^{-1}]_N = 0$, the element xy^{-1} is contained in the image of A_m . Since δA_m is contained in B_m , $(xy^{-1})^\delta$ is contained in the image of B_m . In particular, we have $(xy^{-1})^\delta \in \mathcal{P}_0/p^N$, and we obtain $x^\delta \in \mathcal{P}_0/p^N$. Combine this result with Lemma 3.1, we obtain the lemma. \square

We define the R_N -submodule $\bar{\mathcal{F}}'_N$ of $(F^\times/p^N)_\chi$ by

$$\bar{\mathcal{F}}'_N := \{x^{|\Delta_p|^6} \mid x \in \bar{\mathcal{F}}_N\}.$$

By the first row of the diagram (8), we obtain the following corollary.

Corollary 7.5. *The order of the kernel of $[\cdot]_{m,N,\chi}: \mathcal{F}'_N \longrightarrow \mathcal{J}/p^N$ is finite.*

Proof. Let $\mathcal{P}_0 \subseteq (F^\times/\mathcal{O}_F^\times) \otimes \mathbb{Z}_p$ be as in the proof of Lemma 7.4. We denote the inverse image of \mathcal{P}_0 by the natural homomorphism

$$F^\times \otimes \mathbb{Z}_p \longrightarrow \mathcal{P} = (F^\times/\mathcal{O}_F^\times) \otimes \mathbb{Z}_p$$

by $\tilde{\mathcal{P}}_0$. Note that the kernel of the natural homomorphism

$$\text{pr}_{\mathcal{P}_0,N}: \tilde{\mathcal{P}}_0/p^N \longrightarrow \mathcal{P}_0/p^N$$

coincides with the image of \mathcal{O}_F^\times/p^N in $\tilde{\mathcal{P}}_0/p^N$. So, $\text{Ker pr}_{\mathcal{P}_0}$ has finite order. The top row of the diagram (8) implies that the kernel of the homomorphism

$$[\cdot]'_{m,N}: \mathcal{P}_0/p^N \longrightarrow (J/p^N)_\chi$$

induced by the natural homomorphism $[\cdot]_{m,N}: F^\times/p^N \longrightarrow (J/p^N)_\chi$ has finite order. Then, the kernel of the composite map

$$[\cdot]'_{m,N} \circ \text{pr}_{\mathcal{P}_0,N}: \tilde{\mathcal{P}}_0/p^N \longrightarrow (J/p^N)_\chi$$

is finite.

Let us consider the commutative diagram

$$(9) \quad \begin{array}{ccc} (\tilde{\mathcal{P}}_0/p^N)_\chi & \xrightarrow{([\cdot]_{m,N}' \circ \text{pr}_{\mathcal{P}_0,N})_\chi} & (J/p^N)_\chi \\ \downarrow \iota_1 & & \downarrow \iota_2 \\ \mathcal{F}/p^N & \xrightarrow{\iota_{\mathcal{F},N}} \bar{\mathcal{F}}_N \xrightarrow{[\cdot]_{m,N,\chi}} & \mathcal{J}/p^N \end{array}$$

of natural homomorphisms. Lemma 3.1 implies that the Coker ι_1 is annihilated by $|\Delta_p|^2$. By Corollary 3.3, it follows that $\text{Ker } \iota_2$ is annihilated by $|\Delta_p|^2$. The finiteness of the kernel of $[\cdot]_{m,N}' \circ \text{pr}_{\mathcal{P}_0,N}$ and Corollary 3.3 imply that the order of $|\Delta_p|^2 \text{Ker } ([\cdot]_{m,N}' \circ \text{pr}_{\mathcal{P}_0,N})_\chi$ is finite. Hence we obtain the corollary by chasing the commutative diagram (9). \square

Let n be an element of \mathcal{N}_N whose prime divisors are in P . We define P_F^n to be the set of all elements of P dividing n . We define J_n to be the subgroup of J generated by P_F^n , and the submodule $\mathcal{J}_{n,N}$ of \mathcal{J}/p^N the image of $J_n \otimes \mathbb{Z}_p$ in \mathcal{J}/p^N . We denote by $\bar{\mathcal{F}}_{n,N}$ the inverse image of $\mathcal{J}_{n,N}$ by $[\cdot]_{m,\chi}: (F^\times/p^N)_\chi \rightarrow \mathcal{J}/p^N$. Then, we define the R_N -submodule $\bar{\mathcal{F}}'_{n,N}$ of $(F^\times/p^N)_\chi$

$$\bar{\mathcal{F}}'_{n,N} := \bar{\mathcal{F}}_{n,N} \cap \bar{\mathcal{F}}'_N.$$

Note that $\bar{\mathcal{F}}'_{n,N}$ is a *finite* R_N -submodule of $(F^\times/p^N)_\chi$ by Corollary 7.5.

For each integer r with $1 \leq r \leq h$, let

$$\bar{\alpha}_{r,N}: \mathcal{J}_{n,N} \longrightarrow R_N$$

be the R_N -homomorphism induced by α_r . We put

$$\begin{aligned} \bar{\alpha}_N &:= (\bar{\alpha}_{s,N})_{s=1}^h: \mathcal{J}_{n,N} \longrightarrow R_N^h, \\ \tilde{\beta}_N &:= (\tilde{\beta}_{s,N})_{s=1}^h: \bar{\mathcal{F}}_N \longrightarrow R_N^h. \end{aligned}$$

Then, we obtain the commutative diagram

$$(10) \quad \begin{array}{ccc} \bar{\mathcal{F}}'_{n,N} & \xrightarrow{[\cdot]_{F,\chi}} & \mathcal{J}_{n,N} \\ \tilde{\beta}_N \downarrow & & \downarrow |\Delta_p|^\varepsilon \cdot \bar{\alpha}_N \\ R_N^h & \xrightarrow{\bar{f}} & R_N^h. \end{array}$$

of R_N -modules.

7.2. Let δ_A be a non-zero element of $\text{ann}_{\Lambda_\chi}(X_{\chi,\text{fin}})$. In this and the next subsection, we write $\bar{\phi}^\ell$ in place of $\bar{\phi}_{m,N,\chi}^\ell$ for simplicity. Here, as in [Ku], we shall take the element $x_{\nu,q} \in (F^\times/p^N)_\chi$ which is defined in Definition 5.5, to translate β_r to homomorphisms of the type $\bar{\phi}^\ell$. Recall the element $x_{\nu,q} \in (F^\times/p^N)_\chi$ is determined by η , q , ν , and $\{w_\ell\}_{\ell|\nu}$. We shall take them as follows.

First, let us take a prime number q by the following way. For each integer r with $1 \leq r \leq h$, we fix a prime number $q_r \in P_{n_r}$. We put $Q := \prod_{r=1}^h q_r \in \mathcal{N}_N$. We fix

a homomorphism $\varphi: E_{\infty, \chi} \rightarrow \Lambda_\chi$ of Λ_χ -modules with pseudo-null cokernel. By the Iwasawa main conjecture, we have

$$\varphi(\bar{C}_{\infty, \chi}) = \det(M_0) \cdot I_\varphi(E; C),$$

where $\bar{C}_{\infty, \chi}$ is the image of $C_{\infty, \chi}$ in $E_{\infty, \chi}$, and $I_\varphi(E; C)$ is an ideal of Λ_χ of finite index. We fix an element $\delta_\varphi \in I_\varphi(E; C)$. Then, we fix a family $(\eta_m)_{m \geq 0} \in C_{\infty, \chi}$ of circular units which is defined by Λ_χ -linear combination of basic circular units, and satisfies $\varphi((\eta_m)_{m \geq 0}) = \delta_\varphi \det(M_0)$. We write

$$\eta_m = \eta_m(1) := \prod_{d \mid f_K} \eta_m^d(1)^{u_d} \times \prod_{j=1}^r \eta_m^{1, a_j}(1)^{v_j} \in F_m^\times$$

where r is a positive integer, a_1, \dots, a_r are integers prime to p , and u_d and v_j are elements of Λ_χ for each positive integers d and i with $d \mid f_K$ and $1 \leq i \leq r$. Here, we assume that r, a_1, \dots, a_r, u_d 's and v_j 's are constant independent of m . As in the previous section, we fix m in this and the next section, and put $\eta := \eta_m$ for simplicity.

Let $\bar{\varphi}_{F, N, \chi}: (E_{\infty, \chi})_{\Gamma_m}/p^N \rightarrow R_N$ be the induced homomorphism by $\bar{\varphi}$. Recall that in the proof of Proposition 4.17, we define $\mathcal{NO} := \mathcal{NO}_{m, N, \chi}$ to be the image of the natural homomorphism

$$(E_{\infty, \chi})_{\Gamma_m}/p^N \longrightarrow (\mathcal{O}_F^\times/p^N)_\chi \longrightarrow (F_m^\times/p^N)_\chi.$$

We fix non-zero elements $\delta_I \in I_{P_\chi^E}$ and $\delta_J \in J_{P_\chi^E}$. By the same argument as in Lemma 4.18, there exists a homomorphism $\psi: \mathcal{NO} \rightarrow R_N$ which makes the diagram

$$\begin{array}{ccc} (C_{\infty, \chi})_{\Gamma_m}/p^N & \longrightarrow & (E_{\infty, \chi})_{\Gamma_m}/p_\chi^N \xrightarrow{(\gamma-1)^{a_\chi} \cdot \delta_I \delta_J |\Delta_p|^4 \cdot \bar{\varphi}_{F, N, \chi}} R_N \\ \downarrow & & \downarrow \\ \mathcal{W}_{m, N, \chi}(1) & \hookrightarrow & \mathcal{NO} \end{array} \quad \begin{array}{c} \nearrow \psi \end{array}$$

commute. By Proposition 6.1, we can take a prime number $q \in \mathcal{S}_N$ satisfying the following two conditions:

- (q1) $q \in P'_{n_1}$, and $q \neq q_1$ if $\Delta_p = 0$;
- (q2) $\mathcal{NO}_{m, N, \chi}$ is contained in the kernel of $[\cdot]_{m, N, \chi}^q$, and for all $x \in \mathcal{NO}_{m, N, \chi}$,

$$\bar{\phi}^q(x) = |\Delta_p|^2 \psi(x).$$

In particular, we have

$$\begin{aligned} \bar{\phi}^q(\eta) &= |\Delta_p|^2 \psi(\eta) \\ &= (\gamma - 1)^{a_\chi} |\Delta_p|^6 \delta_I \delta_J \bar{\varphi}_{m, N, \chi}((\eta_m)_{m \geq 0}) \\ &= |\Delta_p|^6 \delta_I \delta_J \delta_\varphi (\gamma - 1)^{a_\chi/2} \cdot \det(M_0). \end{aligned}$$

Next, let us take ν and $\{w_\ell\}_{\ell \mid \nu}$. First, we consider $\tilde{\beta}_{m_1, N}: \tilde{\mathcal{F}}'_{Qq, N} \rightarrow R_N$. By Proposition 6.1, we can take $\ell_2 \in \mathcal{S}_N$ which splits completely in $F(\mu_q)/\mathbb{Q}$, and satisfies

$\ell_2 \in P'_{n_2}$, $\ell \neq q_2$ and

$$\bar{\phi}^{\ell_2}(x) = |\Delta_p|^2 \cdot \tilde{\beta}_{m_1, N}(x)$$

for all $x \in \bar{\mathcal{F}}'_{Qq, N}$. We put $\nu_1 := 1$.

In the case $i = 1$, we put $\nu := \nu_1 = 1$, and $x_{\nu, q} = x_{1, q} = \kappa_{m, N}(\eta; q) = \kappa(q)$. It follows from Proposition 5.6 (1) and Lemma 7.4 that $x_{1, q}^{\delta_A |\Delta_p|^8}$ is an element of $\mathcal{F}'_{Qq, N}$.

Suppose $i \geq 2$. To take ν and $\{w_\ell\}_{\ell|\nu}$, we choose prime numbers ℓ_r for each r with $2 \leq r \leq i+1$ by induction on r as follows. Let r be an integer satisfying $2 < r \leq i+1$. Suppose that for each s with $2 \leq s \leq r-1$, we have chosen distinct prime numbers $\ell_s \in \mathcal{S}_N$ which splits completely in $F(\mu_{q\nu_{s-1}})/\mathbb{Q}$. We put $\nu_{r-1} := \prod_{s=2}^{r-1} \ell_s$. We consider the R_N -linear homomorphism

$$\tilde{\beta}_{m_{r-1}, N}: \bar{\mathcal{F}}'_{Qq\nu_{r-1}, N} \longrightarrow R_N.$$

Applying Proposition 6.1, we can take $\ell_r \in \mathcal{S}_N$ which splits completely in $F(\mu_{q\nu_{r-1}})/\mathbb{Q}$, and satisfies the following conditions:

- (x1) $\ell_r \in P'_{n_r}$, and $\ell_r \neq q_r$ if $\Delta_p = 0$;
- (x2) there exists $b_r \in (F^\times \otimes \mathbb{Z}_p)_\chi$ such that $(b_r)_{F, \chi} = (\ell_{r, F} - |\Delta_p| \cdot q_{r, F})_\chi$ and $\bar{\phi}^{\ell_s}(b_r) = 0$ for any s with $2 \leq s < r$;
- (x3) $\bar{\phi}^{\ell_r}(x) = |\Delta_p|^2 \cdot \tilde{\beta}_{m_{r-1}, N}(x)$ for any $x \in \bar{\mathcal{F}}'_{Qq\nu_{r-1}, N}$.

Thus, we have taken $\ell_2, \dots, \ell_{i+1}$, and we put $\nu := \nu_i = \prod_{r=2}^i \ell_r \in \mathcal{N}_N$. For each r with $2 \leq r \leq i$, we put

$$w_{\ell_r} := -\phi^{\ell_r}(b_r) \in R_N \otimes H_{\ell_r},$$

and we obtain $x_{\nu, q} \in (F^\times/p^N)_\chi$. It follows from Proposition 5.6 (1) and Lemma 7.4 that $x_{\nu, q}^{\delta_A |\Delta_p|^8}$ is an element of $\bar{\mathcal{F}}'_{Qq\nu, N}$. Note that $q\nu$ is *well-ordered*.

7.3. In this subsection, we observe two homomorphism α and β by using $x_{\nu, q}$, and describe $\det(M_i)$ in R_N . First, we prepare the following lemma.

Lemma 7.6 (cf. [Ku] Lemma 9.2). *Suppose $i \geq 2$. Then,*

- (1) $\tilde{\beta}_{m_{r-1}, N}(x_{\nu, q}^{\delta_A |\Delta_p|^{10}}) = 0$ for all r with $2 \leq r \leq i$;
- (2) $\bar{\alpha}_{j, N}([x_{\nu, q}]_{m, N, \chi}) = 0$ for any $j \neq n_1, \dots, n_i$.

Proof. The second assertion (2) of the above lemma is clear by Proposition 5.6 (1).

Let us prove the first assertion. We define an element $y_r \in (F^\times/p^N)_\chi$ by

$$y_r = x_{\nu, q} \prod_{s=r}^i b_s^{\bar{\phi}^{\ell_s}(x_{\nu/\ell_s, q})}$$

for any r satisfying $2 \leq r \leq i$. Note that we have $\bar{\alpha}_N([b_r]_{m, N, \chi}) = 0$ for any r satisfying $2 \leq r \leq i$ since we have $(b_r)_{F, \chi} = (\ell_{r, F} - |\Delta_p| \cdot q_{r, F})_\chi$ and $\ell_r \in P'_{n_r}$. By definition of β , we have $\beta(b_r) = 0$ for any r satisfying $2 \leq r \leq i$. So, we have

$$\tilde{\beta}(x_{\nu, q}^{\delta_A |\Delta_p|^8}) = \tilde{\beta}(y_r^{\delta_A |\Delta_p|^8})$$

for any r with $2 \leq r \leq i$. Let us show $\tilde{\beta}_{m_{r-1},N}(y_r^{\delta_A|\Delta_p|^8}) = 0$ for any integer r satisfying $2 \leq r \leq i$. By Proposition 5.6 (1), we have $[y_r]_{F,N,\chi} \in J_{Qq\nu_{r-1}}$. Then, by Lemma 7.4, we have $y_r^{\delta_A|\Delta_p|^8} \in \mathcal{F}_{Qq\nu_{r-1},N}$. Therefore, we obtain

$$\delta_A |\Delta_p|^8 \bar{\phi}^{\ell_r}(y_r) = |\Delta_p|^2 \tilde{\beta}_{m_{r-1},N}(y_r^{\delta_A|\Delta_p|^8})$$

by the condition (x3). Since $\bar{\phi}^{\ell_r}(b_s) = 0$ for all integers s satisfying $r+1 \leq s \leq i$ by the condition (x2), we have

$$\bar{\phi}^{\ell_r}(y_r) = \bar{\phi}^{\ell_r}(x_{\nu,q} b_r^{\bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q})}).$$

By Proposition 5.6 (3), we have

$$\begin{aligned} \bar{\phi}^{\ell_r}(x_{\nu,q} b_r^{\bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q})}) &= \bar{\phi}^{\ell_r}(x_{\nu,q}) + \bar{\phi}^{\ell_r}(b_r^{\bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q})}) \\ &= -\bar{\phi}^{\ell_r}(b_r) \bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q}) + \bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q}) \bar{\phi}^{\ell_r}(b_r) \\ &= 0. \end{aligned}$$

Therefore, we obtain

$$\tilde{\beta}_{m_{r-1},N}(x_{\nu,q}^{\delta_A|\Delta_p|^{10}}) = |\Delta_p|^2 \tilde{\beta}_{m_{r-1},N}(y_r^{\delta_A|\Delta_p|^8}) = \delta_A |\Delta_p|^8 \bar{\phi}^{\ell_r}(y_r) = 0.$$

□

The goal of this subsection is the following proposition.

Proposition 7.7 (cf. [Ku] p.44). *The following equalities of elements contained in R_N holds.*

(1) *We have*

$$\begin{aligned} \delta_A |\Delta_p|^{10} \det(M) \bar{\phi}^{\ell_2}(x_{1,q}) \\ = \pm |\Delta_p|^{20} (\gamma - 1)^{a_\chi} \delta_A \delta_I \delta_J \varepsilon \det(M_1) \bar{\phi}_{F,N,\chi}((\eta_m)_{m \geq 0}). \end{aligned}$$

(2) *We have*

$$\delta_A |\Delta_p|^{10} \det(M_{r-1}) \bar{\phi}^{\ell_{r+1}}(x_{\nu_r,q}) = \pm |\Delta_p|^{14} \delta_A \varepsilon \det(M_r) \bar{\phi}^{\ell_r}(x_{\nu_{r-1},q})$$

for any r with $2 \leq r \leq i$.

The signs \pm in (1) and (2) do not depend on m .

Proof. For each r satisfying $1 \leq r \leq i$ we put

$$\begin{aligned} \mathbf{x}^{(r)} &:= \tilde{\beta}_N(x_{\nu_r,q}^{|\Delta_p|^{10}\delta_A}) \in R_N^h; \\ \mathbf{y}^{(r)} &:= |\Delta_p| \varepsilon \bar{\alpha}_N(x_{\nu_r,q}^{|\Delta_p|^{10}\delta_A}) \in R_N^h, \end{aligned}$$

and regard them as column vectors. Then, by the commutative diagram (10), we have $\mathbf{y}^{(r)} = M \mathbf{x}^{(r)}$ in R_N^h .

We first prove the assertion (1) of the above proposition. Note that $\delta_A |\Delta_p|^2$ times of $x_{1,q} = \kappa(q)$ is an element of $\mathcal{F}_{q,N}$, and we have

$$\begin{aligned} \mathbf{y}^{(1)} &= |\Delta_p|^{12} \delta_A \varepsilon [\kappa(q)_\chi]_{F,N,\chi}^q \mathbf{e}_{n_1}^{(m)} \\ &= |\Delta_p|^{12} \delta_A \varepsilon \bar{\phi}^q(\eta_m) \mathbf{e}_{n_1}^{(m)} \\ &= |\Delta_p|^{14} \delta_A \varepsilon \psi(\eta_m) \\ &= |\Delta_p|^{18} (\gamma - 1)^{a_\chi} \delta_A \delta_I \delta_J \varepsilon \cdot \bar{\varphi}_{F,N,\chi}((\eta_m)_{m \geq 0}). \end{aligned}$$

Note that the first equation follows from the condition (q1) and definition of α , the second one follows from Proposition 5.3 (2), and the third one follows from the condition (q2). Let \widetilde{M} be the matrix of cofactors of M . Multiplying the both sides of $\mathbf{y}^{(1)} = M\mathbf{x}^{(1)}$ by \widetilde{M} , and comparing the m_1 -st components, we obtain

$$\begin{aligned} &(-1)^{n_1+m_1} |\Delta_p|^{18} (\gamma - 1)^{a_\chi} \delta_A \delta_I \delta_J \varepsilon \det(M_1) \bar{\varphi}_{F,N,\chi}((\eta_m)_{m \geq 0}) \\ &= \det(M) \tilde{\beta}_{m_1,N}(x_{1,q}^{|\Delta_p|^{10} \delta_A}). \end{aligned}$$

By condition (x3) for ℓ_2 , we have

$$|\Delta_p|^2 \tilde{\beta}_{m_1,N}(x_{1,q}^{|\Delta_p|^{10} \delta_A}) = |\Delta_p|^{10} \delta_A \bar{\phi}^{\ell_2}(x_{1,q}),$$

and the assertion (1) follows.

Next, we assume $i \geq 2$, and we shall prove Proposition 7.7 (2). The proof is essentially the same as the proof of assertion (1). It is sufficient to prove the assertion when $r = i$. We write $\mathbf{x} = \mathbf{x}^{(i)}$ and $\mathbf{y} = \mathbf{y}^{(i)}$. Let $\mathbf{x}' \in R_N^{h-i+1}$ be the vector obtained from \mathbf{x} by eliminating the m_j -th rows for $j = 1, \dots, i-1$, and \mathbf{y}' the vector obtained from \mathbf{y} by eliminating the n_k -th rows for $k = 1, \dots, i-1$. Since the m_r -th rows of \mathbf{x} are 0 for all r with $1 \leq r \leq i-1$ by Lemma 7.6 (1), we have $\mathbf{y}' = M_{i-1} \mathbf{x}'$. We assume the m'_i -th component of \mathbf{x}' corresponds to the m_i -th component of \mathbf{x} , and the n'_i -th component of \mathbf{y}' corresponds to the n_i -th component of \mathbf{y} . By Lemma 7.6 (2) and Proposition 5.6 (2), we have

$$\mathbf{y}' = |\Delta_p|^{12} \delta_A \varepsilon \bar{\phi}^{\ell_i}(x_{\nu_{i-1},q}) \mathbf{e}_{n'_i}^{(m)},$$

where $(\mathbf{e}_i^{(m)})_{i=1}^{h-i+1}$ denotes the standard basis of R_N^{h-i+1} .

Let \widetilde{M}_{i-1} be the matrix of cofactors of M_{i-1} . Multiplying the both sides of $\mathbf{y}' = M_{i-1} \mathbf{x}'$ by \widetilde{M}_{i-1} , and comparing the m'_i -th components, we obtain

$$(-1)^{n'_i+m'_i} |\Delta_p|^{12} \delta_A \varepsilon \det(M_i) \bar{\phi}^{\ell_i}(x_{\nu_{i-1},q}) = \det(M_{i-1}) \tilde{\beta}_{m_i,N}(x_{\nu,q}^{|\Delta_p|^{10} \delta_A}).$$

By condition (x3) for ℓ_{i+1} , and since $x_{\nu,q}^{\delta}$ is an element of $\mathcal{F}_{Q\nu,N}$, we have

$$|\Delta_p|^2 \tilde{\beta}_{m_i,N}(x_{\nu,q}^{|\Delta_p|^{10} \delta_A}) = |\Delta_p|^{10} \delta_A \bar{\phi}^{\ell_{i+1}}(x_{\nu,q}).$$

Here, the proof of Proposition 7.7 is complete. \square

7.4. Now let us prove Theorem 7.1. It is convenient to use the following notion of convergence.

Definition 7.8. A sequence $(a_m)_{m \geq 0} \in \prod_{m \geq 0} R_{F_m, N_m, \chi}$ is said to *converge* to

$$b = (b_m)_{m \geq 0} \in \varprojlim_{m \geq 0} R_{F_m, N_m, \chi} = \Lambda_\chi$$

if for each m , there exists an integer L_m such that the image of $a_{m'}$ in $R_{F_m, N_m, \chi}$ coincides to $b_m \in R_{F_m, N_m, \chi}$ for any $m' \geq L_m$.

Proof of Theorem 7.1 . Here, we vary m . In this subsection, we denote the element

$$\bar{\phi}^{\ell_{r+1}}(x_{\nu_r, q}) \in R_N = (\mathbb{Z}/p^{N_m})[\text{Gal}(F_m/F_0)]_\chi$$

defined in §6.2 by $\bar{\phi}^{\ell_{r+1}}(x_{\nu_r, q})_m$. By induction on r , let us prove that the sequence $(\bar{\phi}^{\ell_{r+1}}(x_{\nu_r, q})_m)_{m \geq 0}$ converges to

$$\pm |\Delta_p|^{6+4r} (\gamma - 1)^{a_\chi} \delta_I \delta_J \delta_\varphi \varepsilon^r \det(M_r) \in \Lambda_\chi$$

in the sense of Definition 7.8 for any integer r satisfying $0 \leq r \leq i$.

First, we consider the equality

$$\begin{aligned} & |\Delta_p|^{10} \delta_A \det(M) \cdot \bar{\phi}^{\ell_2}(x_{1, q}) \\ &= \pm |\Delta_p|^{20} (\gamma - 1)^{a_\chi} \delta_A \delta_I \delta_J \varepsilon \det(M_1) \bar{\varphi}_{F, N, \chi}((\eta_m)_{m \geq 0}). \end{aligned}$$

Since the right hand side converges to

$$\pm |\Delta_p|^{20} (\gamma - 1)^{a_\chi} \delta_A \delta_I \delta_J \delta_\varphi \varepsilon \det(M_1) \det(M)$$

and $|\Delta_p|^{10} \delta_A \det(M)$ is non-zero element, it follows that $(\bar{\phi}^{\ell_2}(x_{1, q})_m)_{m \geq 0}$ converges to

$$\pm |\Delta_p|^{10} (\gamma - 1)^{a_\chi} \delta_I \delta_J \delta_\varphi \varepsilon \det(M_1).$$

(Note the sign \pm does not depend on m , see Proposition 7.7).

Next, we assume that the sequence $(\bar{\phi}^{\ell_r}(x_{\nu_{r-1}, q})_m)_{m \geq 0}$ converges to

$$\pm |\Delta_p|^{2+4r} (\gamma - 1)^{a_\chi} \delta_I \delta_J \delta_\varphi \varepsilon^{r-1} \det(M_{r-1}).$$

Then, the right hand side of

$$|\Delta_p|^{10} \delta_A \det(M_{r-1}) \bar{\phi}^{\ell_{r+1}}(x_{\nu_r, q}) = \pm |\Delta_p|^{14} \delta_A \varepsilon \det(M_r) \bar{\phi}^{\ell_r}(x_{\nu_{r-1}, q})$$

converges to

$$\pm |\Delta_p|^{16+4r} (\gamma - 1)^{a_\chi} \delta_A \delta_I \delta_J \delta_\varphi \varepsilon^r \det(M_r) \det(M_{r-1})$$

Since we take $\det(M_{r-1}) \neq 0$, the sequence $(\bar{\phi}^{\ell_{r+1}}(x_{\nu_r, q})_m)_{m \geq 0}$ converges to

$$\pm |\Delta_p|^{6+4r} (\gamma - 1)^{a_\chi} \delta_I \delta_J \delta_\varphi \varepsilon^r \det(M_r).$$

By induction, in particular, we conclude $(\bar{\phi}^{\ell_{i+1}}(x_{\nu, q})_m)$ converges to

$$\pm |\Delta_p|^{6+4i} (\gamma - 1)^{a_\chi} \delta_I \delta_J \delta_\varphi \varepsilon^i \det(M_i).$$

Since $(x_{\nu,q})_m$ is contained in an R_N -submodule of $(F^\times/p^N)_\chi$ generated by the set $\bigcup_{e|q\nu} \mathcal{W}_{m,N,\chi}(e)$ with $\epsilon(q\nu) = i$, we have $\bar{\phi}^{\ell_{i+1}}(x_{\nu,q})_m \in \mathfrak{C}_{i,F_m,N_\chi}$ for all $m \in \mathbb{Z}_{\geq 0}$. Hence we have

$$\pm |\Delta_p|^{6+4i} (\gamma - 1)^{a_\chi} \delta_I \delta_J \delta_\varphi \varepsilon^i \det(M_i) \in \mathfrak{C}_{i,\chi}.$$

This completes the proof of theorem. \square

8. THE HIGHER CYCLOTOMIC IDEALS AND MAZUR-RUBIN THEORY

In this section, we assume that the extension degree of K/\mathbb{Q} is coprime to p , and fix a character $\chi \in \widehat{\Delta}$ satisfying $\chi(p) \neq 1$. We denote by \mathcal{O} the \mathbb{Z}_p -algebra isomorphic to \mathcal{O}_χ with trivial $G_\mathbb{Q}$ -action. We identify the ring \mathcal{O} (resp. $\mathcal{O}[[\Gamma]]$) with \mathcal{O}_χ (resp. Λ_χ) when we ignore the action of $G_\mathbb{Q}$. In particular, we sometimes regard $\mathfrak{C}_{i,\chi}$ as an ideal of $\mathcal{O}[[\Gamma]]$, and $R_{m,N,\chi}$ as a quotient ring of $\mathcal{O}[[\Gamma]]$. In this section, we complete the proof of Theorem 1.1 in §1.

Theorem 8.1 (Theorem 1.1). *We assume that the extension degree of K/\mathbb{Q} is prime to p . Let $\chi \in \widehat{\Delta}$ be a character satisfying $\chi(p) \neq 1$. Then, we have*

$$\text{Fitt}_{\Lambda_\chi, i}(X_\chi) \sim \mathfrak{C}_{i,\chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$. Moreover, we have

$$\text{ann}_{\Lambda_\chi}(X_{\chi, \text{fin}}) \text{Fitt}_{\Lambda_\chi, i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$.

In the previous section, we have already proved

$$\text{ann}_{\Lambda_\chi}(X_{\chi, \text{fin}}) \text{Fitt}_{\Lambda_\chi, i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$$

for any $i \geq 0$. In order to complete the proof of the theorem, we have to show

$$\mathfrak{C}_{i,\chi} \prec \text{Fitt}_{\Lambda_\chi, i}(X_\chi)$$

for any $i \in \mathbb{Z}_{\geq 0}$. It is sufficient to show the following theorem.

Theorem 8.2. *Let i be a positive integer, and $\mathfrak{P} \subset \mathcal{O}[[\Gamma]] = \Lambda_\chi$ a prime ideal of height one containing $\text{Fitt}_{\Lambda_\chi, i}(X_\chi)$. We define two integers $\alpha_i(\mathfrak{P})$ and $\beta_i(\mathfrak{P})$ by*

$$\begin{aligned} \text{Fitt}_{\Lambda_\chi, \mathfrak{P}, i}(X_\chi \otimes_{\mathcal{O}[[\Gamma]]} \Lambda_{\chi, \mathfrak{P}}) &= \mathfrak{P}^{\alpha_i(\mathfrak{P})} \Lambda_{\chi, \mathfrak{P}}, \\ \mathfrak{C}_{i,\chi} \Lambda_{\chi, \mathfrak{P}} &= \mathfrak{P}^{\beta_i(\mathfrak{P})} \Lambda_{\chi, \mathfrak{P}}. \end{aligned}$$

Then, we have $\beta_i(\mathfrak{P}) \geq \alpha_i(\mathfrak{P})$.

In the rest of this section, we prove Theorem 8.2. The key of the proof is comparison between the higher cyclotomic ideals $\mathfrak{C}_{i,\chi}$ defined in this paper and the theory of Kolyvagin systems, which is established by Mazur and Rubin in [MR].

8.1. In the first three subsections, we will review some results on Kolyvagin systems briefly. Here, we recall the notion of Kolyvagin systems in general setting.

Let (R, \mathfrak{m}) be a noetherian complete local ring whose residue field R/\mathfrak{m} is a finite field of characteristic p . We call a triple (T, \mathcal{F}, Σ) which consists of the following data a Selmer structure (cf. [MR] Definition 2.1.1):

- a finite set Σ of places of \mathbb{Q} containing p and ∞ ;
- a free R -module T of finite rank with continuous $G_{\mathbb{Q}}$ -action unramified outside $\Sigma \setminus \{\infty\}$.
- a local condition

$$\mathcal{F} = \{H_{\mathcal{F}}^1(\mathbb{Q}_v, T) \subseteq H^1(\mathbb{Q}_v, T)\}_{v: \text{place of } \mathbb{Q}}$$

on T satisfying

$$H_{\mathcal{F}}^1(\mathbb{Q}_v, T) = H_f^1(\mathbb{Q}_v, T) = H^1(\mathbb{Q}_v^{\text{unr}}/\mathbb{Q}_v, T)$$

for all finite places $v \notin \Sigma$.

We define the Selmer group for a Selmer structure (T, \mathcal{F}, Σ) by

$$H_{\mathcal{F}}^1(\mathbb{Q}, T) := \text{Ker} \left(H^1(\mathbb{Q}, T) \longrightarrow \prod_v \frac{H^1(\mathbb{Q}_v, T)}{H_{\mathcal{F}}^1(\mathbb{Q}_v, T)} \right),$$

where in the product, v runs through all places of \mathbb{Q} .

Let (T, \mathcal{F}, Σ) be a Selmer structure. For each prime number $\ell \notin \Sigma$, we define

$$H_{\text{tr}}^1(\mathbb{Q}_{\ell}, T) := \text{Ker} \left(H^1(\mathbb{Q}_{\ell}, T) \longrightarrow H^1(\mathbb{Q}_{\ell}(\mu_{\ell}), T) \right).$$

Then, we have a direct decomposition $H^1(\mathbb{Q}_{\ell}, T) = H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, T) \oplus H_{\text{tr}}^1(\mathbb{Q}_{\ell}, T)$, so the natural projection

$$H_{\text{tr}}^1(\mathbb{Q}_{\ell}, T) \longrightarrow H_s^1(\mathbb{Q}_{\ell}, T) := H^1(\mathbb{Q}_{\ell}, T)/H_{\mathcal{F}}^1(\mathbb{Q}_{\ell}, T)$$

is an isomorphism. For each square-free product n of finite numbers of primes not contained in Σ , we define a new local condition

$$H_{\mathcal{F}(n)}^1(\mathbb{Q}_v, T) := \begin{cases} H_{\mathcal{F}}^1(\mathbb{Q}_v, T) & \text{if } v \nmid n; \\ H_{\text{tr}}^1(\mathbb{Q}_v, T) & \text{if } v \mid n. \end{cases}$$

Let I be an ideal of R . Then, we denote the image of $H_{\mathcal{F}}^1(\mathbb{Q}_v, T)$ in $H^1(\mathbb{Q}_v, T/IT)$ by $H_{\mathcal{F}}^1(\mathbb{Q}_v, T/IT)$ for any place v of \mathbb{Q} .

In order to define the Kolyvagin systems, we have to construct two homomorphisms of Galois cohomology groups, the “localization” map and the “finite-singular comparison” map. Let (T, \mathcal{F}, Σ) be a Selmer structure over R . For each prime number $\ell \notin \Sigma$, we define

$$P_{\ell}(x) := \det_R(1 - \text{Fr}_{\ell} x \mid T),$$

where $\text{Fr}_{\ell} \in G_{\mathbb{Q}}$ is an arithmetic Frobenius element. Note that the polynomial $P_{\ell}(x)$ is well-defined since the action of $G_{\mathbb{Q}}$ on T is unramified at ℓ . Let I_{ℓ} be the ideal of

R generated by $\ell - 1$ and $P_\ell(1)$. Take a square-free product $n := \ell_1 \times \cdots \times \ell_r$, where ℓ_i is a prime number not contained in Σ for $i = 1, \dots, r$. Then, we define an ideal

$$I_n = \sum_{i=1}^r I_{\ell_i}.$$

Recall that we defined $H_\ell := \text{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q}) \simeq \mathbb{F}_\ell^\times$. By [MR] Definition 1.2.2 and Definition 2.2.1, we can construct a canonical map

$$\phi_\ell^{\text{fs}}: H_f^1(\mathbb{Q}_{\ell_i}, T/I_n T) \longrightarrow H_s^1(\mathbb{Q}_{\ell_i}, T/I_n T) \otimes_{\mathbb{Z}} H_\ell$$

called the finite-singular comparison map for each integer i with $1 \leq i \leq r$. (See loc. cit. for construction and detail of ϕ_ℓ^{fs} .) Note that if $R = \mathcal{O}/p^N$ and $T = \mu_{p^N} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\chi^{-1}}$, then the natural localization map

$$(\cdot)_{\ell,s}: H_{\mathcal{F}(n\ell)}^1(\mathbb{Q}, T/I_{n\ell} T) \otimes_{\mathbb{Z}} H_{n\ell} \longrightarrow H_s^1(\mathbb{Q}_\ell, T/I_{n\ell} T) \otimes_{\mathbb{Z}} H_{n\ell},$$

and the composite map

$$\begin{aligned} H_{\mathcal{F}(n)}^1(\mathbb{Q}, T/I_n T) \otimes_{\mathbb{Z}} H_n &\longrightarrow H_f^1(\mathbb{Q}_\ell, T/I_{n\ell} T) \otimes_{\mathbb{Z}} H_n \\ &\xrightarrow{\phi_\ell^{\text{fs}} \otimes 1} H_s^1(\mathbb{Q}_\ell, T/I_{n\ell} T) \otimes_{\mathbb{Z}} H_{n\ell}, \end{aligned}$$

coincide with the map

$$[\cdot]_{0,N,\chi}^\ell \otimes 1: (F_0^\times/p^N)_\chi \otimes_{\mathbb{Z}} H_{n\ell} \longrightarrow R_{0,N,\chi} \otimes_{\mathbb{Z}} H_{n\ell}$$

defined in Definition 5.1 and

$$\phi_{0,N,\chi}^\ell \otimes 1: (F_0^\times/p^N)_\chi \otimes_{\mathbb{Z}} H_n \longrightarrow R_{0,N,\chi} \otimes_{\mathbb{Z}} H_{n\ell}$$

defined in Definition 5.2 respectively for any $\ell \in \mathcal{S}_N$ and any $n \in \mathcal{N}_N$ with $(\ell, n) = 1$.

Let us recall the definition of Kolyvagin systems. Let (T, \mathcal{F}, Σ) be a Selmer structure over R and \mathcal{P} be a set of rational primes disjoint from Σ . We denote the set of all square-free products of \mathcal{P} by $\mathcal{N}(\mathcal{P})$. (Note $1 \in \mathcal{N}(\mathcal{P})$.) We call such a triple $(T, \mathcal{F}, \mathcal{P})$ a Selmer triple.

Definition 8.3. A Kolyvagin system for a Selmer triple $(T, \mathcal{F}, \mathcal{P})$ is a family of cohomology classes

$$\kappa = \{\kappa_n \in H_{\mathcal{F}(n)}^1(\mathbb{Q}, T/I_n T) \otimes_{\mathbb{Z}} H_n\}_{n \in \mathcal{N}(\mathcal{P})}$$

satisfying

$$(\kappa_{n\ell})_{\ell,s} = \phi_\ell^{\text{fs}}(\kappa_n) \quad \text{in} \quad H_s^1(\mathbb{Q}_\ell, T/I_{n\ell} T) \otimes_{\mathbb{Z}} H_{n\ell}$$

for any $n \in \mathcal{N}(\mathcal{P})$ and any $\ell \in \mathcal{P}$ satisfying $n\ell \in \mathcal{N}(\mathcal{P})$. We denote the set of all Kolyvagin systems for $(T, \mathcal{F}, \mathcal{P})$ by $\text{KS}(T, \mathcal{F}, \mathcal{P})$.

Let $(T, \mathcal{F}, \mathcal{P})$ be a Selmer triple over R . We naturally regard an abelian group

$$T^* := \text{Hom}_{\mathbb{Z}_p}(T, \mu_{p^\infty})$$

as an $R[G_\mathbb{Q}]$ -module. For all finite places v of \mathbb{Q} , we denote the orthogonal complement of $H_{\mathcal{F}}^1(\mathbb{Q}_v, T)$ in the sense of the local duality pairing

$$H^1(\mathbb{Q}_v, T) \times H^1(\mathbb{Q}_\ell, T^*) \longrightarrow H^2(\mathbb{Q}_v, \mu_{p^\infty}) \simeq \mathbb{Q}_p/\mathbb{Z}_p$$

by $H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_v, T^*)$. If $v = \infty$, we have $H^1(\mathbb{Q}_v, T) = 0$ and $H^1(\mathbb{Q}_v, T^*) = 0$ since p is odd. In certain good situations, Kolyvagin systems for the triple $(T, \mathcal{F}, \mathcal{P})$ describe the structure of the Selmer group

$$H_{\mathcal{F}^*}^1(\mathbb{Q}, T^*) = \text{Ker} \left(H^1(\mathbb{Q}, T^*) \longrightarrow \prod_v \frac{H^1(\mathbb{Q}_v, T^*)}{H_{\mathcal{F}^*}^1(\mathbb{Q}_v, T^*)} \right),$$

where in the product, v runs through all places of \mathbb{Q} .

For $r \in \mathbb{Z}_{\geq 0}$, we denote by \mathcal{P}_r the set of all prime numbers ℓ not contained in Σ satisfying both of the following conditions:

- $T/(\mathfrak{m}^r T + (\text{Fr}_\ell - 1)T)$ is a free R -module of rank one;
- $I_\ell \subseteq \mathfrak{m}^r$.

Then, we have $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$ for any positive integer i .

Definition 8.4. Let $(T, \mathcal{F}, \mathcal{P})$ be a Selmer triple over R . Then, we define

$$\overline{\text{KS}}(T, \mathcal{F}, \mathcal{P}) := \varprojlim_k \varinjlim_j \text{KS}(T/\mathfrak{m}^k T, \mathcal{F}, \mathcal{P} \cap \mathcal{P}_j).$$

Note that $\overline{\text{KS}}(T, \mathcal{F}, \mathcal{P})$ matches Euler system arguments better than $\text{KS}(T, \mathcal{F}, \mathcal{P})$. (See Theorem 8.6.)

Remark 8.5. We have a natural homomorphism

$$\text{KS}(T, \mathcal{F}, \mathcal{P}) \longrightarrow \overline{\text{KS}}(T, \mathcal{F}, \mathcal{P}).$$

of R -modules. It may not be either injective or surjective in general. (See [MR] p. 21.) But, later, it will turn out that this homomorphism is an isomorphism in the special case which we treat in our paper (cf. Proposition 8.9.)

8.2. In this subsection, we review the notation on Selmer structures over complete discrete valuation rings and Iwasawa algebras.

Now, we set a Selmer structure over complete discrete valuation rings. Let R be the integer ring of a finite extension field of \mathbb{Q}_p , and T a free R -module of finite rank with continuous $G_{\mathbb{Q}}$ -action unramified at all but finitely many primes. Throughout this subsection, we assume that T satisfies conditions (H.0)-(H.4) in [MR] §3.5. We set the local condition \mathcal{F}_{can} for T by

$$H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}_v, T) := \begin{cases} H_f^1(\mathbb{Q}_v, T) & \text{if } v \text{ is a finite place prime to } p; \\ H^1(\mathbb{Q}_p, T) & \text{if } v = p; \\ H^1(\mathbb{R}, T) = 0 & \text{if } v = \infty, \end{cases}$$

where for finite places v prime to p , the local condition $H_f^1(\mathbb{Q}_v, T)$ is in the sense of Bloch-Kato. Namely, we define

$$H_f^1(\mathbb{Q}_v, T) = \text{Ker} \left(H^1(\mathbb{Q}_v, T) \longrightarrow H^1(\mathbb{Q}_v^{\text{unr}}, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \right).$$

Note that the triple $(T, \mathcal{F}, \mathcal{P}_1)$ satisfies all of the conditions (H.0)-(H.6) in [MR] §3.5.

Next, we set a Selmer structure over Iwasawa algebra. Let R be the integer ring of a finite extension field of \mathbb{Q}_p , and (T, \mathcal{F}, Σ) a Selmer structure over R . We define the $R[[\Gamma]]$ -module \mathbf{T} with a continuous $R[[\Gamma]]$ -linear $G_{\mathbb{Q}}$ -action by

$$\mathbf{T} := T \otimes_R R[[\Gamma]],$$

and the local condition \mathcal{F}_{Λ} on \mathbf{T} by

$$H_{\mathcal{F}_{\Lambda}}^1(\mathbb{Q}_v, \mathbf{T}) := H^1(\mathbb{Q}_v, \mathbf{T}).$$

Let \mathbb{Q}_{Σ} be the maximal extension of \mathbb{Q} unramified outside Σ . Then, by standard arguments, we have

$$H^1(\mathbb{Q}, \mathbf{T}) = H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q}, \mathbf{T}) \simeq \varprojlim H^1(\mathbb{Q}_{\Sigma}/\mathbb{Q}_n, T),$$

where the limit in the most right term is defined by the projective system with respect to corestriction maps. (See, for example [MR] Lemma 5.3.1.) In particular, the local condition \mathcal{F}_{Λ} coincides the finite local condition f . So, the triple $(\mathbf{T}, \mathcal{F}_{\Lambda}, \Sigma)$ is a Selmer structure.

In the last of this subsection, we recall the relation between Euler systems and Kolyvagin systems. Let R be the integer ring of a finite extension field of \mathbb{Q}_p , and $(T, \mathcal{F}, \mathcal{P})$ a Selmer triple over R . We define the field \mathcal{K} to be the composite field of \mathbb{Q}_{∞} and $\mathbb{Q}(\mu_n)$ for all positive integers n satisfying $(n, \ell) = 1$ for any prime number $\ell \in \Sigma$. Let \mathfrak{N}_{Σ} be the ideal of \mathbb{Z} defined by the square free product of the prime numbers contained in Σ . We denote the set of Euler systems for $(T, \mathcal{K}/\mathbb{Q}, \mathfrak{N}_{\Sigma})$ in the sense of [Ru5] by $\text{ES}(T, \Sigma)$.

Theorem 8.6 (Theorem 3.2.4 in [MR]). *Let $(T, \mathcal{F}, \mathcal{P})$ be a Selmer triple satisfying the conditions (H.0)-(H.6) in [MR] §3.5. Assume the following two conditions:*

- *the R -module $T/(\text{Fr}_{\ell} - 1)T$ is cyclic for any $\ell \in \mathcal{P}$;*
- *the action of $\text{Fr}_{\ell}^{p^k} - 1$ on T is injective for any $\ell \in \mathcal{P}$ and any $k \in \mathbb{Z}_{\geq 0}$.*

Then, there exists a R -linear map

$$\text{ES}(T, \Sigma) \longrightarrow \overline{\text{KS}}(T, \mathcal{F}, \mathcal{P}); \quad c = \{c_F\}_{F \subseteq \mathcal{K}} \longmapsto \kappa(c) := \{\kappa(c)_n\}_n$$

satisfying $\kappa(c)_1 = c_{\mathbb{Q}}$ in $H^1(\mathbb{Q}, T)$. This map is constructed by using Kolyvagin derivative classes of Euler systems. For the detail of the construction of this map, see [MR] Appendix A.

8.3. In this subsection, we recall some results (specialized for our purpose) on Kolyvagin system proved in [MR] §§5.2-5.3. (In [MR] §5.3, they treat only the case of $\mathcal{O}_{\chi} = \mathbb{Z}_p$, but we can prove similar results for general \mathcal{O}_{χ} by similar arguments.)

Recall that \mathcal{O} is a \mathbb{Z}_p -algebra isomorphic to \mathcal{O}_{χ} with trivial $G_{\mathbb{Q}}$ -action. We define a \mathcal{O} -module T_{χ} by

$$T_{\chi} := \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\chi}^{-1}.$$

We define a set Σ of places of \mathbb{Q} by

$$\Sigma := \{p, \infty\} \cup \{\ell \mid \ell \text{ ramifies in } K/\mathbb{Q}\},$$

and consider the Selmer structure $(T_\chi, \mathcal{F}_{\text{can}}, \Sigma)$. We fix $\mathcal{P} := \mathcal{P}_1$. Note that by Kummer theory, we have

$$H^1(\mathbb{Q}, T/p^N T) = (F_0^\times/p^N)_\chi$$

for any positive integer N . Since we assume $\chi(p) \neq 1$, we have

$$H^1(\mathbb{Q}_p, T_\chi^*) = \left(\bigoplus_{\lambda|p} \text{Hom}(G_{F_{0,\lambda}}, \mathbb{Q}_p/\mathbb{Z}_p) \right)_{\chi^{-1}} = 0,$$

so by global class field theory, we have

$$\begin{aligned} & H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, T_\chi^*) \\ &= \text{Ker} \left(\text{Hom}(G_{F_0}, \mathbb{Q}_p/\mathbb{Z}_p)_{\chi^{-1}} \longrightarrow \prod_{\ell \neq p, \infty} \left(\bigoplus_{\lambda|\ell} \text{Hom}(G_{F_{0,\lambda}^{\text{unr}}}, \mathbb{Q}_p/\mathbb{Z}_p) \right)_{\chi^{-1}} \right) \\ & \quad \cap \text{Ker} \left(\text{Hom}(G_{F_0}, \mathbb{Q}_p/\mathbb{Z}_p)_{\chi^{-1}} \longrightarrow \left(\bigoplus_{\lambda|p} \text{Hom}(G_{F_{0,\lambda}}, \mathbb{Q}_p/\mathbb{Z}_p) \right)_{\chi^{-1}} \right) \\ &= \text{Ker} \left(\text{Hom}(G_{F_0}, \mathbb{Q}_p/\mathbb{Z}_p)_{\chi^{-1}} \longrightarrow \prod_{\ell \neq \infty} \left(\bigoplus_{\lambda|\ell} \text{Hom}(G_{F_{0,\lambda}^{\text{unr}}}, \mathbb{Q}_p/\mathbb{Z}_p) \right)_{\chi^{-1}} \right) \\ &= \text{Hom}(A_{F_0, \chi}, \mathbb{Q}_p/\mathbb{Z}_p). \end{aligned}$$

Recall we define $\Lambda_\chi := \mathcal{O}_\chi[[\Gamma_0]]$. We define

$$\mathbf{T}_\chi := T_\chi \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma]] = \Lambda_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1).$$

Note that by local duality, we have $H^2(\mathbb{Q}_p, T_\chi^*) = 0$, and by (the limit of) Poitou-Tate exact sequence, we obtain

$$X_\chi = \text{Hom}(H_{\mathcal{F}_\Lambda}^1(\mathbb{Q}, T^*), \mathbb{Q}_p/\mathbb{Z}_p) = H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathbf{T}).$$

We denote define a set Σ_Λ of hight one prime ideals of the ring $\mathcal{O}[[\Gamma]] = \Lambda_\chi$ by

$$\Sigma_\Lambda := \{ \mathfrak{P} \in \text{Spec}(\Lambda_\chi) \mid \text{ht} \mathfrak{P} = 1 \text{ and } \text{char}_{\Lambda_\chi}(X_\chi) \subseteq \mathfrak{P} \} \cup \{ p\Lambda_\chi \}.$$

Let $\mathfrak{Q} \subseteq \mathcal{O}[[\Gamma]] = \Lambda_\chi$ be a prime ideal of hight one not contained in Σ_Λ . We denote the integral closure of $\mathcal{O}[[\Gamma]]/\mathfrak{Q}$ by $S_\mathfrak{Q}$. Note that $S_\mathfrak{Q}$ is a complete discrete valuation ring of mixed characteristic $(0, p)$ whose residue field is finite. We fix a uniformizer of $S_\mathfrak{Q}$ by π .

Here, let us consider the Selmer triple

$$(\mathbf{T}_\chi \otimes_{\mathcal{O}[[\Gamma]]} S_\mathfrak{Q} = T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}, \mathcal{F}_{\text{can}}, \mathcal{P} := \mathcal{P}_1).$$

Note that this Selmer triple satisfies conditions (H.0)-(H.6) in [MR] §3.5. We have the following lemma.

Lemma 8.7 ([MR] Lemma 5.3.16). *Let $\mathfrak{Q} \subseteq \mathcal{O}[[\Gamma]]$ be a prime ideal of hight one not contained in Σ_Λ . Then, we have*

$$\chi(T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}) = \text{rank}_{\mathcal{O}}(T_\chi^-) = 1$$

where $\chi(T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q})$ is the core rank of $T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}$ in the sense of [MR] Definition 5.2.4.

Remark 8.8. Note that the original statement of Lemma 5.3.16 in [MR] treats only the case of $\mathcal{O} = \mathbb{Z}_p$, but the similar proof works for general \mathcal{O} .

By Lemma 8.7 and [MR] Proposition 5.2.9, we obtain the following corollary.

Corollary 8.9. *Let $\mathfrak{Q} \subseteq \mathcal{O}[[\Gamma]]$ be a prime ideal of height one not contained in Σ_Λ . Then, the natural homomorphism*

$$\mathrm{KS}(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}) \longrightarrow \overline{\mathrm{KS}}(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$$

is an isomorphism.

For each positive integer N , we define

$$\mathcal{S}_N(\mathfrak{Q}) := \{\ell \in \mathcal{S}_N \mid \mathrm{Fr}_\ell \text{ acts trivially on } T \otimes_{\mathcal{O}} S_{\mathfrak{Q}}/p^N S_{\mathfrak{Q}}\},$$

and we denote the set of all well-ordered square-free products of $\mathcal{S}_N(\mathfrak{Q})$ by $\mathcal{N}_N^{\mathrm{w.o.}}(\mathfrak{Q})$. Note that we have $\mathcal{S}_{N'} \subseteq \mathcal{S}_N(\mathfrak{Q})$ for any sufficiently large N' . Consider the R -linear map

$$\mathrm{ES}(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}) \longrightarrow \overline{\mathrm{KS}}(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}) = \mathrm{KS}(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$$

in Theorem 8.6. Then, by construction of this map (cf. [MR] pp. 80-81.), we obtain the following proposition.

Proposition 8.10. *Take $c \in \mathrm{ES}(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \mathcal{K}/\mathbb{Q}, \mathcal{P})$ and $n \in \mathcal{N}_N^{\mathrm{w.o.}}(\mathfrak{Q})$. Let $\kappa_{0,N}(c; n)$ be the Kolyvagin derivative class of c at n . Then, we have*

$$\kappa(c)_n = \kappa_{0,N}(c; n)$$

in $H^1(\mathbb{Q}, T_\chi \otimes_{\mathcal{O}} (S_{\mathfrak{Q}}/p^N))$.

Recall that for each element $n := \ell_1 \times \cdots \times \ell_r \in \mathcal{N}(\mathcal{P})$, we denote the number of prime divisors of n by $\epsilon(n) := r$. For any non-zero Kolyvagin system $\kappa = \{\kappa_n\} \in \mathrm{KS}(T_\chi S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$, we define

$$\partial_i(\kappa; \mathfrak{Q}) := \max\{j \in \mathbb{Z}_{\geq 0} \mid \pi^j H_{\mathcal{F}(n)}(\mathbb{Q}, T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}/I_n) \text{ for all } n \in \mathcal{N}(\mathcal{P}) \text{ with } \epsilon(n) = i\}$$

for any non-negative integer i . We also define

$$\partial_i(\mathfrak{Q}) := \min\{\partial_i(\kappa; \mathfrak{Q}) \mid \kappa = \{\kappa_n\} \in \mathrm{KS}(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})\}.$$

Note that we have $e_i(\mathfrak{Q}) := \partial_i(\mathfrak{Q}) - \partial_{(i+1)}(\mathfrak{Q}) \geq 0$ for any $i \in \mathbb{Z}_{\geq 0}$, and $\partial_i(\mathfrak{Q}) = 0$ for sufficiently large i . Note that the core rank of $T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}$ is one by Lemma 8.7, and we can check the Selmer triple $(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ satisfies all conditions required in [MR] Theorem 5.2.12, we obtain the following result.

Theorem 8.11 (a special case of [MR] Theorem 5.2.12). *Let $\mathfrak{Q} \subseteq \mathcal{O}[[\Gamma]]$ be a prime ideal of height one not contained in Σ_Λ . We fix a uniformizer $\pi_{\mathfrak{Q}}$ of $S_{\mathfrak{Q}}$. Then, we have an isomorphism*

$$H_{\mathcal{F}_\Lambda^*}^1(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}})^*) \simeq \bigoplus_{i \geq 0} S_{\mathfrak{Q}}/\pi_{\mathfrak{Q}}^{e_i(\mathfrak{Q})} S_{\mathfrak{Q}}$$

of $S_{\mathfrak{p}}$ -modules. In other words, we have

$$\mathrm{Fitt}_{S_{\mathfrak{Q}}, i}(H_{\mathcal{F}_{\mathrm{can}}^*}^1(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}})^*)) = \pi_{\mathfrak{Q}}^{\partial_i(\mathfrak{Q})} S_{\mathfrak{Q}}$$

for any $i \in \mathbb{Z}_{\geq 0}$.

8.4. In this section, we treat the results on higher Fitting ideals of $A_{0,\chi}$. By Theorem 8.11 and usual Euler system arguments (without Kurihara's elements), we obtain the following theorem.

Theorem 8.12. *Assume the extension degree of K/\mathbb{Q} is prime to p . Let $\chi \in \widehat{\Delta}$ be a character satisfying $\chi(p) \neq 1$. Then, we have*

$$\text{Fitt}_{\mathcal{O}_\chi/p^N, i}(A_{0,\chi}) = \mathfrak{C}_{i,0,N,\chi}$$

for any non-negative integer i and sufficiently large integer N .

Proof. The order of χ is prime to p , so p is a prime element of the discrete valuation ring \mathcal{O}_χ . Since $A_{0,\chi}$ is a finitely generated torsion \mathcal{O}_χ -module, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_\chi^r \xrightarrow{f} \mathcal{O}_\chi^r \xrightarrow{g} A_{0,\chi} \longrightarrow 0,$$

of \mathcal{O}_χ -modules, where the matrix M_f associated to f for the standard basis $(\mathbf{e}_j)_{j=1}^r$ of \mathcal{O}_χ^r is a diagonal matrix

$$M_f := \begin{pmatrix} p^{d_1} & & & \\ & p^{d_2} & & \\ & & \ddots & \\ & & & p^{d_r} \end{pmatrix}$$

satisfying $d_1 \geq d_2 \geq \cdots \geq d_r$.

We fix an integer N satisfying $p^N \geq \#A_{0,\chi}$. First, let us show the inequality $\text{Fitt}_{\mathcal{O}_\chi/p^N, i}(A_{0,\chi}) \supseteq \mathfrak{C}_{i,0,N,\chi}$. Let

$$\eta = \{\eta_m(n)_\chi \in (F_m(\mu_n)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi\}_{m,n}$$

be an Euler system of circular units defined by a Λ_χ -linear combination of basic circular units, $n \in \mathcal{N}_N^{\text{w.o.}}$ satisfying $\epsilon(n) \leq i$, and

$$f: (F_0^\times/p^N)_\chi \longrightarrow R_{0,N,\chi} = \mathcal{O}_\chi/p^N$$

an arbitrary homomorphism of $R_{0,N,\chi}$ -modules. Then, from Theorem 8.11 for the prime ideal $(\gamma - 1)\Lambda_\chi$ and Proposition 8.10, it follows that $\kappa_{0,N,\chi}(n, \eta)_\chi$ is a $p^{\sum_{j=i+1}^r d_j}$ -power of some element in $(F_0^\times/p^N)_\chi$. This implies

$$f(\kappa_{0,N}(\eta; n)_\chi) \in p^{\sum_{j=i+1}^r d_j}(\mathcal{O}_\chi/p^N) = \text{Fitt}_{\mathcal{O}_\chi, i}(A_{0,\chi})(\mathcal{O}_\chi/p^N),$$

and we obtain $\text{Fitt}_{\mathcal{O}_\chi/p^N, i}(A_{0,\chi}) \supseteq \mathfrak{C}_{i,0,N,\chi}$.

Note that the inequality $\text{Fitt}_{\mathcal{O}_\chi, i}(A_{0,\chi}) \subseteq \mathfrak{C}_{i,0,N,\chi}$ follows from the usual Euler system argument. (See, for example, the arguments in [Ru2] §4.) We sketch the proof of this inequality briefly. Let N be a sufficiently integer. Note that any circular unit in F_0 extends to an Euler system defined by a Λ_χ -linear combination of basic circular units since we assume $\chi(p) \neq 1$ (cf. Remark 4.6). Fix an Euler system

$$\eta = \{\eta_m(n)_\chi \in (F_m(\mu_n)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi\}_{m,n}$$

circular units defined by a Λ_χ -linear combination of basic circular units, and assume that the circular unit $\eta_0(1)$ generates the free \mathcal{O}_χ -module $C_{0,\chi} = (C_0 \otimes \mathbb{Z}_p)_\chi$ of rank

one. Recall that $E_{0,\chi} := (\mathcal{O}_{F_0}^\times \otimes \mathbb{Z}_p)_\chi$ is a free \mathcal{O}_χ -module of rank one. We fix an isomorphism

$$\psi_0: E_{0,\chi}/p^N \longrightarrow R_{0,N,\chi} = \mathcal{O}_\chi/p^N$$

of \mathcal{O}_χ -modules and a prime number ℓ_1 whose ideal class $[\ell_1]_{F_0,\chi}$ in $A_{0,\chi}$ coincides with $g(\mathbf{e}_1)$ and satisfying

$$\phi_{m,N,\chi}^{\ell_1} |_{E_{0,\chi}/p^N} = \psi_0.$$

(Note that Proposition 6.1 ensures the existence of such a prime number ℓ_1 .) By the arguments in [Ru2] §4 combined with Proposition 6.1, we can inductively take prime numbers $\ell_1, \dots, \ell_{r+1} \in \mathcal{S}_N$ homomorphisms

$$\psi_j: (F_0^\times/p^N)_\chi \longrightarrow R_{0,N,\chi} = \mathcal{O}_\chi/p^N \quad (j = 1, \dots, r)$$

satisfying the following conditions.

- $[\ell_j]_{F_0,\chi} = g(\mathbf{e}_j)$ in $A_{0,\chi}$ for any integer j with $1 \leq j \leq r$.
- The integer $n_j := \prod_{\nu=1}^j \ell_\nu$ is well-ordered any integer j with $1 \leq j \leq r$.
- $p^{d_j-1} \psi_j(\kappa_{0,N}(\eta; n_j)) = \psi_{j-1}(\kappa_{0,N}(\eta; n_{j-1}))_\chi$ for any integer j with $1 \leq j \leq r$. Here, we put $n_0 := 1$.
- The restriction of $\phi_{0,N,\chi}^{\ell_j}$ on $\mathcal{W}_{0,N,\chi}(n_j)$ coincides with ψ_{j-1} any integer j with $1 \leq j \leq r$.

Then, we obtain

$$p^{\sum_{j=1}^{i-1} d_j} \psi_i(\kappa_{0,N}(\eta; n_i)_\chi) = p^{\sum_{j=1}^{i-2} d_j} \psi_{i-1}(\kappa_{0,N}(\eta; n_{i-1})_\chi) = \dots = \psi_0(\eta).$$

By [Ru2] Theorem 4.2 (see [MW] Theorem 1.10.1 or [Ru5] Corollary 3.2.4 for general cases), there exists a unit $u \in \mathcal{O}_\chi^\times$ such that

$$\psi_0(\eta) = u \# A_{0,\chi} = u p^{\sum_{j=1}^r d_j}.$$

Therefore, we obtain

$$\mathfrak{C}_{0,N,\chi} \subseteq \psi_i(\kappa_{0,N}(\eta; n_i)_\chi) R_{0,N,\chi} = p^{\sum_{j=i+1}^r d_j} R_{0,N,\chi}.$$

This completes the proof. \square

Remark 8.13. Fix a pseudo-isomorphism

$$X_\chi \longrightarrow \bigoplus_{j=1}^r \Lambda_\chi / f_j \Lambda_\chi,$$

where f_1, \dots, f_r are non-unit and non-zero elements of Λ_χ satisfying $f_r | \dots | f_2 | f_1$. Then, by similar argument to the proof of the inequality $\text{Fitt}_{\mathcal{O}_\chi, i}(A_{0,\chi}) \subseteq \mathfrak{C}_{i,0,N,\chi}$, we can prove rough estimates

$$(11) \quad \text{Fitt}_{\Lambda_\chi, i}(X_\chi) \prec \mathfrak{C}_{i,\chi}$$

without using Kurihara's elements. In this argument, we have to use the argument of [Ru2] §5 and the Iwasawa main conjecture instead of the argument of [Ru2] §4 and [Ru2] Theorem 4.2. Note that when we apply such arguments without Kurihara's elements, we have to ignore error factors (11) completely. So, Theorem 7.1, which is proved by Euler arguments via Kurihara's elements, is stronger than results obtained by usual arguments without Kurihara's elements.

Note that Theorem 8.12 implies that for any non-negative integer i and any two integers N and N' satisfying $N' \geq N > 0$, the image of $\mathfrak{C}_{0,N',\chi}$ in $R_{0,N,\chi}$ coincides with $\mathfrak{C}_{0,N,\chi}$. Combining this fact and the second assertion of Corollary 4.14, we obtain the following corollary immediately.

Corollary 8.14. *Assume the extension degree of K/\mathbb{Q} is prime to p . Let i be a non-negative integer, and $\chi \in \widehat{\Delta}$ a character satisfying $\chi(p) \neq 1$. Then, the following holds.*

- (1) *The image of $\mathfrak{C}_{i,\chi}$ in $R_{0,N,\chi}$ coincides with the ideal $\mathfrak{C}_{i,0,N,\chi}$ for any positive integer N .*
- (2) *The image of $\mathfrak{C}_{i,\chi}$ in $R_{0,\chi} := \mathbb{Z}_p[\text{Gal}(F_0/\mathbb{Q})]_\chi$ coincides with the ideal $\mathfrak{C}_{i,F_0,\chi} := \varprojlim_N \mathfrak{C}_{i,0,N,\chi}$.*

We put $\mathfrak{m} := p\Lambda_\chi + (\gamma - 1)\Lambda_\chi$. Note we have the natural isomorphism

$$X_\chi/\mathfrak{m}X_\chi \simeq A_{0,\chi}/p$$

by Proposition 3.10. So, the least cardinality of generators of the Λ_χ -module X_χ coincides to that of the \mathcal{O}_χ/p -module $A_{0,\chi}$ by Nakayama's lemma. Hence the following corollary follows from Remark 2.3, Theorem 8.12 and Corollary 8.14.

Corollary 8.15. *Let K/\mathbb{Q} and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. Let r be a non-negative integer. Then, the following two properties are equivalent.*

- (1) *The least cardinality of generators of the Λ_χ -module X_χ is r .*
- (2) *$\mathfrak{C}_{r-1,\chi} \neq \Lambda_\chi$ and $\mathfrak{C}_{r,\chi} = \Lambda_\chi$.*

Example 8.16. In general, the computation of the higher cyclotomic ideals $\mathfrak{C}_{i,\chi}$ is hard. But in a certain very special case, we determine the higher cyclotomic ideals explicitly and prove that they coincide with the higher Fitting ideals. Let $p = 3$ and $K := \mathbb{Q}(\sqrt{257})$. Then, we have $F_0 = K$. We take a unique non-trivial character $\chi \in \widehat{\Delta}$. In this case, Greenberg proved that $A_{n,\chi}$ is a cyclic group of order 3 for any $n \geq 0$. (See [Gree2] §7.) So, we have $X_\chi = X_{\chi,\text{fin}} \simeq \Lambda_\chi/(3, \gamma - 1)$ by Proposition 3.10, and we obtain $\text{Fitt}_{\Lambda_\chi,0}(X_\chi) = (3, \gamma - 1)$ and $\text{Fitt}_{\Lambda_\chi,i}(X_\chi) = \Lambda_\chi$ for $i \geq 1$. Note that $\gamma - 1$ annihilates $X_{\chi,\text{fin}}$, so Corollary 7.2 implies that the element $\gamma - 1$ belongs to $\mathfrak{C}_{i,\chi}$ for any $i \geq 0$. Since both $\text{Fitt}_{\Lambda_\chi,i}(X_\chi)$ and $\mathfrak{C}_{i,\chi}$ contain the ideal $(\gamma - 1)\Lambda_\chi$ for any $i \geq 0$, we deduce that

$$\mathfrak{C}_{i,\chi} = \text{Fitt}_{\Lambda_\chi,i}(X_\chi) = \begin{cases} (3, \gamma - 1) & \text{if } i = 0 \\ \Lambda_\chi & \text{if } i > 0 \end{cases}$$

from Theorem 8.12 and Corollary 8.14.

8.5. In this subsection, we prove Theorem 8.2. Here, we fix a positive integer i and a height one prime ideal \mathfrak{P} of $\mathcal{O}[[\Gamma]]$ containing $\text{Fitt}_{\Lambda_\chi}(X_\chi)$. In particular, we have $\mathfrak{P} \neq (p)$. For simplicity, we put $\alpha := \alpha_i(\mathfrak{P})$ and $\beta := \beta_i(\mathfrak{P})$. We define a non-negative integer s by

$$p^s = (S_{\mathfrak{P}} : \mathcal{O}[[\Gamma]]/\mathfrak{P}).$$

We regard $\mathcal{O}[[\Gamma]]$ as the ring $\mathcal{O}[[T]]$ of formal power series by the isomorphism $\mathcal{O}[[\Gamma]] \simeq \mathcal{O}[[T]]$ defined by $\gamma \mapsto 1 + T$. Let $f(T) \in \mathcal{O}_\chi[T]$ be the Weierstrass polynomial generating the fixed prime ideal \mathfrak{P} of $\mathcal{O}[[\Gamma]] = \mathcal{O}_\chi[[T]]$. For any positive integer k , we put

$$f_k(T) := f(T) + p^k$$

and let \mathfrak{P}_k be the principal ideal of $\Lambda_\chi = \mathcal{O}_\chi[[T]]$ generated by $f_k(T)$. We need the following lemma (cf. [MR] p. 66).

Lemma 8.17. *There exists a positive integer $N(\mathfrak{P})$ satisfying the following properties.*

- (1) *The ideal \mathfrak{P}_k is a prime for any $k \geq N(\mathfrak{P})$.*
- (2) *The ideal \mathfrak{P}_k is not contained in Σ_Λ for any $k \geq N(\mathfrak{P})$.*
- (3) *The residue ring $\mathcal{O}[[\Gamma]]/\mathfrak{P}_k$ is (non-canonically) isomorphic to $\mathcal{O}[[\Gamma]]/\mathfrak{P}$ as \mathcal{O} -algebra for any $k \geq N(\mathfrak{P})$.*
- (4) *The action of $\text{Fr}_\ell^{p^m} - 1$ on $T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k}$ is injective for any $\ell \in \mathcal{P}$, any $m \geq 0$ and any $k \geq N(\mathfrak{P})$.*

Proof. The arguments in [MR] p. 66 implies that there exists an integer $N'(\mathfrak{P})$ such that the conditions (1)-(3) in the lemma holds for any integer k satisfying $k \geq N'(\mathfrak{P})$. (See loc. cit. for detail.) So, it is sufficient to show that the fourth condition holds for any sufficiently large integer k . We denote the cyclotomic character by

$$\chi^{\text{cyc}}: \Gamma \longrightarrow 1 + p\mathbb{Z}_p \hookrightarrow \mathcal{O}^\times.$$

Let k be an integer satisfying $k \geq N(\mathfrak{P})$. The natural projection induces a continuous character

$$\rho_k: \Gamma \longrightarrow (\mathcal{O}[[\Gamma]]/\mathfrak{P}_k)^\times.$$

Note that the action of $\text{Fr}_\ell^{p^m} - 1$ on $T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k}$ is *not* injective for some $\ell \in \mathcal{P}$ and some $m \geq 0$ if and only if the order of the character $\chi^{\text{cyc}}\rho_k$ is finite.

We denote the order of p -power torsion part of $(\mathcal{O}[[\Gamma]]/\mathfrak{P}_k)^\times$ by p^ν . Assume that the character $\chi^{\text{cyc}}\rho_k$ has finite order. The image of $\chi^{\text{cyc}}\rho_k$ is contained in $(\mathcal{O}[[\Gamma]]/\mathfrak{P}_k)^\times$, so the character $\chi^{\text{cyc}}\rho_k$ is annihilated by p^ν . In particular, we have

$$\rho_k(\gamma^{p^\nu}) = (\chi^{\text{cyc}})^{-p^\nu}(\gamma)$$

in $\mathcal{O}[[\Gamma]]/\mathfrak{P}_k$. This implies the polynomial

$$(1 + T)^{p^\nu} - (\chi^{\text{cyc}})^{-p^\nu}(\gamma) \in \mathcal{O}[T]$$

is divisible by the monic polynomial $f_k(T)$. Obviously, such a situation occurs for only finitely many k , so the condition (4) holds for any sufficiently large integer k . \square

Definition 8.18. Let M be an integer, and $\{x_k\}_{k \in \mathbb{Z}_{\geq M}}$ and $\{y_k\}_{k \in \mathbb{Z}_{\geq M}}$ sequences of real numbers. We write $x_k \succ y_k$ if and only if $\liminf_{k \rightarrow \infty} (x_k - y_k) \neq -\infty$. We write $x_N \sim y_N$ if and only if $x_k \succ y_k$ and $y_k \succ x_k$. In other words, we write $x_k \sim y_k$ if and only if $|x_k - y_k|$ is bounded independent of N .

We denote the ramification index of $\text{Frac}(S_{\mathfrak{P}})/\mathbb{Q}_p$ by $e_{\mathfrak{P}}$, and the extension degree of the residue field of \mathcal{O}_{χ} over \mathbb{F}_p by f_{χ} . Let us recall the observations in [MR] p. 66. Let d be a non-negative integer. Then, we have

$$\mathfrak{P}_k + \mathfrak{P}^d = \mathfrak{P}_k + p^{de_{\mathfrak{P}}k} \mathcal{O}[[\Gamma]]$$

for any sufficiently large integer k . So, we obtain the natural isomorphism

$$(\mathcal{O}[[\Gamma]]/\mathfrak{P}^d) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \simeq S_{\mathfrak{P}_k}/(\pi_k)^{de_{\mathfrak{P}}k}$$

of $S_{\mathfrak{P}_k}$ -algebras for any sufficiently large k . Moreover, we obtain the following lemma from the observations in [MR] p. 66. (See [MR] loc. cit. for the proof.)

Lemma 8.19. *Let M be a finitely generated torsion $\mathcal{O}[[\Gamma]]$ -module, and*

$$E := \bigoplus_{j=0}^r \mathcal{O}[[\Gamma]]/\mathfrak{P}^{d_j} \oplus \bigoplus_{j'=0}^{r'} \mathcal{O}[[\Gamma]]/(g_{j'}(T)^{e_{j'}})$$

an elementary $\mathcal{O}[[\Gamma]]$ -module, where d_j and $e_{j'}$ are positive integers, and $g_{j'}(T)$ is a Weierstrass polynomial in $\mathcal{O}_{\chi}[T]$ prime to $f(T)$ for any integer i and j' . Suppose that the $\mathcal{O}[[\Gamma]]$ -module M is pseudo-isomorphic to E . Then, there exists a sequence

$$\left\{ \iota_k : M \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow \bigoplus_{j=0}^r S_{\mathfrak{P}_k}/(\pi_k)^{d_j e_{\mathfrak{P}}k}; S_{\mathfrak{P}_k}\text{-linear} \right\}_{k \geq N(\mathfrak{P})}$$

of homomorphisms such that the orders of the kernel and cokernel of ι_k are finite for any $k \geq N(\mathfrak{P})$, and bounded by a constant independent of k .

Then, we immediately obtain the following Corollary 8.20 of Lemma 8.19 combined with Lemma 2.6. This corollary plays an important role in this section.

Corollary 8.20. *Let M be a finitely generated torsion $\mathcal{O}[[\Gamma]]$ -module. We define a non-negative integers C by*

$$\text{Fitt}_{\mathcal{O}[[\Gamma]], i}(M \otimes_{\mathcal{O}[[\Gamma]]} \Lambda_{\chi, \mathfrak{P}}) = \mathfrak{P}^C \Lambda_{\chi, \mathfrak{P}}.$$

For each positive integer k with $k \geq N(\mathfrak{P})$, fix a uniformizer π_k of $S_{\mathfrak{P}_k}$, and define a non-negative integer c_k by

$$\text{Fitt}_{S_{\mathfrak{P}_k}, i}(M \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}}) = \pi_k^{c_k} S_{\mathfrak{P}}.$$

Then, we have $c_k \sim C e_{\mathfrak{P}} k$.

Definition 8.21. Let k be a non-negative integer with $k \geq N(\mathfrak{P})$. We define non-negative integers a_k and b_k by

$$\begin{aligned} \pi_k^{a_k} S_{\mathfrak{P}_k} &= \text{Fitt}_{S_{\mathfrak{P}_k}, i}(X_{\chi} \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}), \\ b_k &= \text{length}_{S_{\mathfrak{P}_k}}((\mathcal{O}[[\Gamma]]/\mathfrak{C}_{i, \chi}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}). \end{aligned}$$

By Corollary 8.20, we have $a_k \sim \alpha e_{\mathfrak{P}} k$ and $b_k \sim \beta e_{\mathfrak{P}} k$.

Proposition 8.22 ([MR] Proposition 5.3.14). *Let k be an integer satisfying $k \geq N(\mathfrak{P})$, and*

$$\pi_k : X_{\chi} \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow \text{Hom} \left(H_{\mathcal{F}_{\text{can}}}^1(\mathbb{Q}, (T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})^*), \mathbb{Q}_p/\mathbb{Z}_p \right)$$

a natural homomorphism. Then, the kernel and cokernel of π_k are both finite, and the orders of kernel and cokernel of π_k are bounded by a constant independent of k .

Combining Proposition 8.22 with Theorem 8.11, we obtain the following corollary.

Corollary 8.23. *We have $a_k \sim \partial_i(\mathfrak{P}_k)$.*

For an integer k with $k \geq N(\mathfrak{P})$, we take an integer N'_k satisfying

- $N'_k \geq \partial_i(\mathfrak{P}_k)$, and
- $p^{N'_k} \in \mathfrak{C}_{i,\chi} + \mathfrak{P}_k$.

Note that there exist such an N'_k since the ideal $\mathfrak{C}_{i,\chi} + \mathfrak{P}_k$ has finite index in $\mathcal{O}[[\Gamma]]$. Then, we take $N''_k \geq N'_k$ satisfying

- $\gamma^{p^{N''_k}-1} - 1 \in \mathfrak{P}_k + p^{N''_k} \mathcal{O}[[\Gamma]]$, and
- $\mathcal{S}_{N''_k} \subseteq \mathcal{S}_{N'_k}(\mathfrak{P}_k)$.

We put $m_k := N''_k - 1$.

Proof of 8.2. Now, we shall prove Theorem 8.2. It is sufficient to show $Be_{\mathfrak{P}}k \succ Ae_{\mathfrak{P}}k$. Let k be any positive integer satisfying $k \geq N(\mathfrak{P})$. Then, we have

$$\begin{aligned} \beta e_{\mathfrak{P}}k &\sim b_k = \text{length}_{S_{\mathfrak{P}_k}}((\mathcal{O}[[\Gamma]]/(\mathfrak{C}_{i,\chi} + \mathfrak{P}_k)) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}) \\ &= \text{length}_{S_{\mathfrak{P}_k}}((\mathcal{O}[[\Gamma]]/(\mathfrak{C}_{i,\chi} + \mathfrak{P}_k + p^{N'_k} \mathcal{O}[[\Gamma]])) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}) \\ &= \text{length}_{S_{\mathfrak{P}_k}}((\mathcal{O}[[\Gamma]]/(\mathfrak{C}_{i,\chi} + \mathfrak{P}_k + (p^{N'_k}, \gamma^{p^{m_k}} - 1))) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}) \\ &= \text{length}_{S_{\mathfrak{P}_k}}((R_{m_k, N'_k, \chi}/(\text{the image of } \mathfrak{C}_{i,\chi})) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}) \\ &\geq \text{length}_{S_{\mathfrak{P}_k}}((R_{m_k, N'_k, \chi}/(\text{the image of } \mathfrak{C}_{i, m_k, N''_k, \chi})) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}) \end{aligned}$$

Since the ring $R_{m_k, N'_k, \chi} \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}$ is a quotient of the discrete valuation ring $S_{\mathfrak{P}_k}$, the image of $\mathfrak{C}_{i, m_k, N''_k, \chi}$ in $R_{m_k, N'_k, \chi} \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}$ is a principal ideal. So, there exist

- a circular unit

$$\eta(n_k) = \eta_{m_k}^{(k)}(n_k) := \prod_{d \mid f_K} \eta_{m_k}^d(n)^{u_d} \times \prod_{i=1}^r \eta_{m_k}^{1, a_i}(n)^{v_i} \in F_{m_k}(\mu_{n_k})^\times,$$

where $r \in \mathbb{Z}_{>0}$, u_d and v_i are elements of $\mathbb{Z}[\text{Gal}(F_m/\mathbb{Q})]$ for each positive integers d and i with $d \mid f_K$ and $1 \leq i \leq r$, and a_1, \dots, a_r are integers prime to p ,

- an element $n_k \in \mathcal{N}_{N''_k}^{\text{w.o.}}$,
- a homomorphism $h_k: F_m^\times/p^{N''_k} \longrightarrow R_{m_k, N'_k, \chi}$,

such that the ideal of $R_{m_k, N'_k, \chi} \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}$ is generated by the image of $h(\kappa_{m_k, N''_k}(\eta; n_k))$. Therefore, we obtain

$$(12) \quad Be_{\mathfrak{P}}k \succ \text{length}_{S_{\mathfrak{P}_k}}(R_{m_k, N'_k, \chi}/h(\kappa_{m_k, N''_k}(\eta; n_k))S_{\mathfrak{P}_k}).$$

We denote by $\bar{h}_k: F_m^\times/p^{N'_k} \longrightarrow R_{m_k, N'_k, \chi}$, the $R_{m_k, N'_k, \chi}$ -linear homomorphism induced by h_k .

For a moment, we fix an integer $k \geq N(\mathfrak{P})$, and put $N' := N_k$, $N'' := N'_k$, $m := m_k$, $n = n_k$ and $\bar{h}_k := \bar{h}$ for simplicity. We put

$$N_{H_n} := \sum_{\sigma \in H_n} \sigma \in \mathbb{Z}[H_n].$$

Let $\nu_{H_n}: R_{m, N', \chi} \longrightarrow R_{m, N, \chi}[H_n]$ be the isomorphism of $R_{m, N', \chi}[H_n]$ -module defined by $1 \mapsto N_{H_n}$. Note that $R_{m, N', \chi}[H_n]$ is a injective $R_{m_k, N'_k, \chi}$ -module, so there exist an $R_{m_k, N'_k, \chi}$ -linear homomorphism $\tilde{h}: (F_m(\mu_n)^\times/p^{N'})_\chi \longrightarrow R_{m, N', \chi}[H_n]$ which makes the diagram

$$\begin{array}{ccc} (F_m^\times/p^{N'})_\chi & \xrightarrow{h} & R_{m, N', \chi} \\ \downarrow & & \downarrow \nu_{H_n} \\ (F_m(\mu_n)^\times/p^{N'})_\chi & \xrightarrow{\tilde{h}} & R_{m, N', \chi}[H_n] \end{array}$$

commute.

Note that by Shapiro's lemma, we have a natural isomorphism

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \xrightarrow{\simeq} \varprojlim_{m'} (F_{m'}(\mu_n)^\times/p^{N'})_\chi.$$

Then, by Lemma 4.13 (ii), we obtain the following lemma.

Lemma 8.24. *There exists a homomorphism*

$$\tilde{h}_\infty: H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \longrightarrow \mathcal{O}[[\Gamma]][H_n]/p^{N''} = \varprojlim_{m'} R_{m', N', \chi}[H_n]$$

of $\mathcal{O}[[\Gamma]][H_n]/p^{N'}$ -modules which makes the diagram

$$\begin{array}{ccc} H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) & \xrightarrow{\tilde{h}_\infty} & \mathcal{O}[[\Gamma]][H_n]/p^{N'} \\ \downarrow & & \downarrow \text{mod } (\gamma^{p^m} - 1) \\ (F_m(\mu_n)^\times/p^{N'})_\chi & \xrightarrow{\tilde{h}} & R_{m, N', \chi}[H_n] \end{array}$$

commute.

Recall that we define a non-negative integer s by

$$p^s = (S_{\mathfrak{P}} : \mathcal{O}[[\Gamma]]/\mathfrak{P}).$$

Let us show the following proposition.

Proposition 8.25. *There exists a $S_{\mathfrak{P}_k}[H_n]$ -linear homomorphism*

$$\tilde{h}_{\mathfrak{P}_k, N'}: H^1(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_N})/p^{N'}) \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'}$$

which makes the diagram

$$\begin{array}{ccc} H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) & \xrightarrow{p^{3s}\tilde{h}_\infty} & \mathcal{O}[[\Gamma]][H_n]/p^{N'} \\ \downarrow & & \downarrow \\ H^1(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_N})/p^{N'}) & \xrightarrow{\tilde{h}_{\mathfrak{P}_k}} & S_{\mathfrak{P}_k}[H_n]/p^{N'} \end{array}$$

commute. Here, the vertical maps in this diagram are the natural map.

Proof. We divide the vertical maps in some short steps, and we will construct suitable homomorphisms step by step. Recall that define a non-negative integer s by

$$p^s = (S_{\mathfrak{P}} : \mathcal{O}[[\Gamma]]/\mathfrak{P}).$$

In order to prove the proposition, we need the following Lemma 8.26 and its corollary. (Note that the following lemma is proved by similar arguments to Lemma 3.1, so we omit the proof.)

Lemma 8.26. *Let M be a $\mathcal{O}[[\Gamma]]$ -module. Then, the kernel and the cokernel of the natural $\Lambda_\chi/\mathfrak{P}_k$ -linear map*

$$M \longrightarrow M \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k}$$

is annihilated by p^s .

The following corollary follows from Lemma 8.26 by the similar arguments to Corollary 3.3.

Corollary 8.27. *Let $f: M \longrightarrow N$ be a homomorphism of $\mathcal{O}[[\Gamma]]/\mathfrak{P}_k$ -modules. Consider the $S_{\mathfrak{P}_k}$ -linear map*

$$f \otimes S_{\mathfrak{P}_k}: M \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k} \longrightarrow N \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k}$$

induced by f . Then, the $S_{\mathfrak{P}_k}$ -submodule $p^s \text{Ker}(f \otimes S_{\mathfrak{P}_k})$ of $M \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k}$ is contained in the image of $\text{Ker } f$.

Here, we return to the proof of the proposition. From the exact sequence

$$0 \longrightarrow \mathbf{T}_\chi/p^{N'} \xrightarrow{\times f_k(T)} \mathbf{T}_\chi/p^{N'} \longrightarrow \mathbf{T}_\chi/(p^{N'}\mathbf{T}_\chi + \mathfrak{P}_k\mathbf{T}_\chi) \longrightarrow 0,$$

it follows that the natural homomorphism

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \otimes_{\mathcal{O}[[\Gamma]]} (\mathcal{O}[[\Gamma]]/\mathfrak{P}_k) \longrightarrow H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/(p^{N'}\mathbf{T}_\chi + \mathfrak{P}_k\mathbf{T}_\chi))$$

is injective. So, by Corollary 8.27, the kernel of the $S_{\mathfrak{P}_k}[H_n]$ -linear map

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/(p^{N'}\mathbf{T}_\chi + \mathfrak{P}_k\mathbf{T}_\chi)) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}$$

is annihilated by p^s .

The $\mathcal{O}[[\Gamma]][H_n]$ -linear homomorphism \tilde{h}'_∞ induces an $S_{\mathfrak{P}_k}[H_n]$ -linear homomorphism

$$\tilde{h}^{(0)}: H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'}.$$

For simplicity, we put

$$\overline{\mathbf{T}}_{\chi,k} := \mathbf{T}_{\chi}/(p^{N'}\mathbf{T}_{\chi} + \mathfrak{P}_k\mathbf{T}_{\chi}).$$

Let $\text{Im}_{S_{\mathfrak{P}_k}}$ be the image of $S_{\mathfrak{P}_k}[H_n]$ -linear map

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_{\chi}/p^{N'}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}$$

Then, there exists an $S_{\mathfrak{P}_k}[H_n]$ -linear homomorphism

$$\tilde{h}^{(1)}: \text{Im}_{S_{\mathfrak{P}_k}} \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'}$$

which makes the diagram

$$\begin{array}{ccc} H^1(\mathbb{Q}(\mu_n), \mathbf{T}_{\chi}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} & \xrightarrow{p^s \tilde{h}^{(0)}} & S_{\mathfrak{P}_k}[H_n]/p^{N'} \\ \downarrow & \nearrow \tilde{h}^{(1)} & \\ \text{Im}_{S_{\mathfrak{P}_k}} & & \end{array}$$

commute. Note that $S_{\mathfrak{P}_k}[H_n]/p^{N'}$ is an injective $S_{\mathfrak{P}_k}[H_n]/p^{N'}$ -module since $S_{\mathfrak{P}_k}/p^{N'}$ is a quotient of a complete discrete valuation ring $S_{\mathfrak{P}_k}$ with finite residue field. So, we can extend $\tilde{h}^{(1)}$ to an $S_{\mathfrak{P}_k}[H_n]/p^{N'}$ -linear map

$$\tilde{h}^{(1)}: H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'},$$

and we obtain the commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{Q}(\mu_n), \mathbf{T}_{\chi}/p^{N'}) & \xrightarrow{p^s \tilde{h}_{\infty}} & \mathcal{O}[[\Gamma]][H_n]/p^{N'} \\ \downarrow & & \downarrow \\ H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} & \xrightarrow{\tilde{h}^{(1)}} & S_{\mathfrak{P}_k}[H_n]/p^{N'}. \end{array}$$

Note that $\text{Tor}_1^{\Lambda_{\chi}/\mathfrak{P}_k}(\overline{\mathbf{T}}_{\chi,k}, S_{\mathfrak{P}_k}/(\mathcal{O}[[\Gamma]]/\mathfrak{P}_k))$ and $\overline{\mathbf{T}}_{\chi,k} \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k}/\overline{\mathbf{T}}_{\chi,k}$ are annihilated by p^s , the kernel of the natural homomorphism

$$H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow H^1(\mathbb{Q}(\mu_n), (T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'})$$

is annihilated by p^{2s} . Then, from the injectivity of $S_{\mathfrak{P}_k}[H_n]/p^{N'}$, there exists an $S_{\mathfrak{P}_k}[H_n]/p^{N'}$ -linear map

$$\tilde{h}_{\mathfrak{P}_k}: H^1(\mathbb{Q}(\mu_n), (T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}) \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'},$$

which makes the diagram

$$\begin{array}{ccc} H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} & \xrightarrow{p^{2s} \tilde{h}^{(1)}} & S_{\mathfrak{P}_k}[H_n]/p^{N'} \\ \downarrow & \nearrow \tilde{h}_{\mathfrak{P}_k} & \\ H^1(\mathbb{Q}(\mu_n), (T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}) & & \end{array}$$

commute. The homomorphism $\tilde{h}_{\mathfrak{P}_k}$ is what we want to construct, and this completes the proof of Proposition 8.25. \square

We identify $S_{\mathfrak{p}_k}/p^{N'}$ with $S_{\mathfrak{p}_k}/p^{N'}[H_n]^{H_n}$ as an $S_{\mathfrak{p}_k}/p^{N'}[H_n]$ -module by the isomorphism

$$\nu_{H_n}: S_{\mathfrak{p}_k}/p^{N'} \longrightarrow S_{\mathfrak{p}_k}/p^{N'}[H_n]^{H_n}; \quad 1 \longrightarrow N_{m+1/m},$$

and let

$$h_{\mathfrak{p}_k}: H^1(\mathbb{Q}, (T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{p}_N})/p^{N'}) \longrightarrow S_{\mathfrak{p}_k}/p^{N'}$$

be the homomorphism induced by $\tilde{h}_{\mathfrak{p}_k}$. Note that since we assume the ideal $\mathfrak{P}_k + p^{N'}\mathcal{O}[[\Gamma]]$ contains $\gamma^{p^m} - 1$, the natural homomorphism

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_{\chi}/p^{N'}) \longrightarrow H^1(\mathbb{Q}(\mu_n), (T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{p}_N})/p^{N'})$$

factors through

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_{\chi}/((\gamma^{p^{m'}} - 1)\mathbf{T}_{\chi} + p^{N''}\mathbf{T}_{\chi})) \simeq (F_m(\mu_n)^{\times}/p^{N'})_{\chi}.$$

We denote the image of $H^1(\mathbb{Q}(\mu_n), \mathbf{T}_{\chi}/p^{N'})$ in $(F_m(\mu_n)^{\times}/p^{N'})_{\chi}$ by Im_F . Then, by Proposition 8.25, we obtain the commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{Q}(\mu_n), \mathbf{T}_{\chi}/p^{N'}) & \xrightarrow{p^{3s}\tilde{h}_{\infty}} & \mathcal{O}[[\Gamma]][H_n]/p^{N'} \\ \downarrow & & \downarrow \\ \text{Im}_F & \xrightarrow{p^{3s}\tilde{h}} & R_{m,N',\chi}[H_n] \\ \downarrow & & \downarrow \\ H^1(\mathbb{Q}(\mu_n), (T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{p}_N})/p^{N'}) & \xrightarrow{\tilde{h}_{\mathfrak{p}_k}} & S_{\mathfrak{p}_k}[H_n]/p^{N'} \\ \uparrow & & \uparrow \nu_{H_n} \\ H^1(\mathbb{Q}, (T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{p}_N})/p^{N'}) & \xrightarrow{h_{\mathfrak{p}_k}} & S_{\mathfrak{p}_k}/p^{N'}. \end{array}$$

By the norm compatibility of circular units, we can define the element

$$\eta_{\infty}^{D_n} := (\eta_m(n)^{D_n}) \in H^1(\mathbb{Q}(\mu_n), \mathbf{T}_{\chi}/p^{N'}) = \varprojlim_{m'} (F_{m'}(\mu_n)^{\times}/p^{N'})_{\chi}.$$

In particular, we have

$$\eta_m(n)^{D_n} \in \text{Im}_F = (F_m(\mu_n)^{\times}/p^{N'})_{\chi}.$$

Let \mathcal{O}_S be a ring which is isomorphic to $S_{\mathfrak{p}_k}$ as a \mathcal{O}_{χ} -algebra, and we assume that the Galois group $G_{\mathbb{Q}}$ acts on \mathcal{O}_S trivially. The action of $G_{\mathbf{Q}}$ on $S_{\mathfrak{p}_k}$ defines a continuous character

$$\rho: \Gamma \longrightarrow \mathcal{O}_S^{\times}.$$

We regard both $T_{\chi} \otimes_{\mathcal{O}} \mathcal{O}_S$ and $T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{p}_k}$ as free \mathcal{O}_S -modules of rank one. Let

$$\eta \otimes \rho := \{\eta \otimes \rho_F \in H^1(F, (T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{p}_k}))\}_{F \subset \mathcal{K}}$$

be the Euler system for $(T_{\chi} \otimes_{\mathcal{O}} S_{\mathfrak{p}_k}, \mathcal{K}/\mathbb{Q}, \Sigma)$ which is the twist of the Euler system

$$\eta^{(k)} := \{\eta_{m'}^{(k)}(n') \in H^1(F_{m'}(\mu_{n'}), T_{\chi})\}_{F_{m'}(\mu_{n'}) \subset \mathcal{K}}$$

for $(T_\chi \otimes_{\mathcal{O}} \mathcal{O}_S, \mathcal{K}/\mathbb{Q}, \Sigma)$ by the character ρ in the sense of [Ru5]. Since we assume $n \in \mathcal{N}_{N''}^{\text{w.o.}} \subseteq \mathcal{N}^{\text{w.o.}}(\mathcal{S}_{N'}(\mathfrak{P}_k))$, we can define the Kolyvagin derivative class

$$\kappa_{0,N'}(\eta \otimes \rho; n) \in H^1(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'})$$

of the Euler system $\eta \otimes \rho$, whose image in $H^1(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'})$ coincides with the image of $\eta_m(n)^{D_n}$. By Proposition 8.10, we have

$$\kappa_{0,N'}(\eta \otimes \rho; n) = \kappa(\eta \otimes \rho)_n \in H^1(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}),$$

where $\kappa(\eta \otimes \rho)_n$ is the n -component of the Kolyvagin system defined by the Euler system $\eta \otimes \rho$. Therefore, we obtain

$$\begin{aligned} \beta e_{\mathfrak{P}_k} k &\succ \text{length}_{S_{\mathfrak{P}_k}}(S_{\mathfrak{P}_k}/(p^{N'} S_{\mathfrak{P}_k} + h(\kappa_{m_k, N''}(\eta; n_k)) S_{\mathfrak{P}_k})) \\ &\sim \text{length}_{S_{\mathfrak{P}_k}}(S_{\mathfrak{P}_k}/(p^{N'} S_{\mathfrak{P}_k} + p^{3s} \tilde{h}(\eta_{m_k}(n_k)^{D_{n_k}}) S_{\mathfrak{P}_k})) \\ &= \text{length}_{S_{\mathfrak{P}_k}}(S_{\mathfrak{P}_k}/(p^{N'} S_{\mathfrak{P}_k} + h_{\mathfrak{P}_k}(\kappa(\eta \otimes \rho)_n) S_{\mathfrak{P}_k})) \\ &\geq \min\{\partial_i(\mathfrak{P}_k), N'\} = \partial_i(\mathfrak{P}_k) \sim a_k \\ &\sim \alpha e_{\mathfrak{P}_k} k \end{aligned}$$

Thus, we obtain $\beta \geq \alpha$, and this completes the proof of Theorem 8.2. \square

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